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VIBRATION AND BUCKLING CHARACTERISTICS
OF COMPOSITE CYLINDRICAL PANELS
INCORPORATING THE EFFECTS OF A
HIGHER ORDER SHEAR THEORY

THESIS

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THESIS

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List of Symbols

a	Length in x direction
$A_{mn}, B_{mn}, C_{mn}, E_{mn}, G_{mn}$	Constant Coefficients in Admissible Functions
A_{ij}	Extensional Stiffness Terms
b	Circumferential Length in y Direction
B_{ij}	Coupling Stiffness Terms
D_{ij}	Bending Stiffness Terms
$E_{ij}, F_{ij}, G_{ij}, H_{ij}, I_{ij}$ J_{ij}	Higher Order Stiffness Terms
E_i	Young's Modulus in the ith Direction
$\varepsilon_x, \varepsilon_y, \varepsilon_z$	Normal Strains in x, y, and z directions
ϕ_{mn}	Generalized Admissible Functions
G_{12}, G_{23}, G_{13}	Shear Modulus in 1-2, 2-3, and 1-3 planes
$\gamma_{xy}, \gamma_{yz}, \gamma_{xz}$	Shear Strains in x-y, y-z, and x-z planes
h	Laminate thickness
$I_i, \overline{I}_i, \overline{I}'_i$	Mass Moments of Inertia
ψ_x	Mid Surface Degree of Freedom: Rotation of Cross Section about x axis
ψ_y	Mid Surface Degree of Freedom: Rotation of Cross Section about y axis
k	Constant = $-4/(3h^2)$
$\varkappa_i^j$	Mid surface Curvature Term
L_i, S_i, P_i, R_i	Higher Order Resultant Quantities
m, n	Summation Indicies that Generate the Number of Terms per Galerkin Equation
M, N	Summation Limits for Generating Galerkin Equations
M_i	Moment Resultants

N_i, Q_i	Force Resultants
$\bar{N}_1, \bar{N}_2, \bar{N}_6$	Buckling Loads in x,y, and x-y directions
ν_{ij}	Poisson's Ratio
p, q	Summation Indicies that Generate the Number of Galerkin Equations
Q_{ij}	Reduced Stiffness Terms
$\bar{Q}_{ij}$	Transformed Reduced Stiffness Terms
θ_k	Fiber orientation angle of ply k
R	Radius of Curvature
ρ	Mass Density
S_{ij}	Compliance Terms
σ_x, σ_y	Normal Stress in x and y directions
t_k	Thickness of ply k
t	Time
T	Kinetic Energy
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	Shear Stress in x-y,x-z,y-z Planes
u_o	Mid Surface Displacement in x direction
u	Displacement ir x direction
U	Strain Energy
v_o	Mid Surface Displacement in y direction
v	Displacement in y direction
V	Potential Energy
w	Displacement in z direction
ω	Frequency of Vibration

Abstract

An analytical study is conducted to determine the fundamental frequencies and critical buckling loads for laminated anisotropic circular cylindrical shell panels, including the effects of transverse shear deformation and rotary inertia, by using the Galerkin technique. A linearized form of Sander's shell strain-displacement relations are derived, which include a parabolic distribution of transverse shear strains. The theory is valid for laminate thickness to radius ratios, h/R, of up to 1/5. Higher order constitutive relations are derived for the laminate. A set of five coupled partial differential equations of motion and boundary conditions are derived and then solved using the Galerkin technique. Simply supported and clamped boundary conditions are investigated.

The Galerkin method is tested for convergence to exact solutions. Comparisons with Donnell shell solutions are conducted. The effects of transverse shear deformation and rotary inertia are examined by comparing the results with classical solutions, where applicable. The radius of curvature is varied to determine the effects of membrane and bending coupling.

It is found that the Galerkin technique converges for all panel configurations investigated; additionally, it is found that buckling problems need more terms in the approximations than vibration problems to obtain proper convergence. The theory

compares exactly with the Donnell solutions, which are valid up to $h/R = 1/50$. As expected, as length to thickness ratios are reduced, shear deformation effects significantly lower the natural frequencies and buckling loads. Analysis also shows that rotary inertia effects are very small. Finally, as h/R is varied from 0 (flat plate) tc 1/5 (maximum limit), the frequencies and buckling loads increase due to membrane and bending coupling.

VIBRATION AND BUCKLING CHARACTERISTICS OF COMPOSITE CYLINDRICAL PANELS INCORPORATING THE EFFECTS OF A HIGHER ORDER SHEAR THEORY

I. INTRODUCTION

Advanced composite materials, so named due to their high strength and stiffness to weight ratios, are seeing widespread use in many diverse industries. One of these is the aerospace industry, where complex shell configurations are common structural elements. Structural elements consisting of composite materials offer unique advantages over those made of traditional isotropic materials in that properties can be tailored to meet specific design goals. Optimization of properties through tailoring can reduce the overall weight of a structure, since stiffness and strength are designed only where they are required. A lower weight structure translates into higher performance. (8)

Because of the potentially large spatial variations of stiffness properties in these composite shell structures due to tailoring, three dimensional stress and strain effects become very important. Whereas classical two dimensional assumptions may be valid for an identical shell structure consisting of isotropic materials, they may lead to gross inaccuracies for an

orthotropic construction. (8)

To ensure a structurally strong and stable product, the designer needs to know the buckling and vibration characteristics of the structural elements, along with other important properties. Cylindrical shell panels are a common shell configuration in aerospace structural applications and are one of the few shell elements that may be analyzed analytically, rather than resorting to a finite element numerical approach.

In light of the above, this thesis focuses on the fundamental natural frequencies of vibration and the critical buckling loads of composite circular cylindrical shell panels including the following:

1. Linear displacement and rotations, and linear elastic behavior of cylindrical shells and flat plates.
2. Parabolic transverse shear strain and stress modeling.
3. Bifurcation buckling analysis.
4. Harmonic vibration analysis excluding transients.
5. Analytical solution method using the Galerkin technique.

BACKGROUND

There have been many contributors to the research of this thesis. This section will briefly address previous work related to this research in an approximate order of what occurred historically.

Past research has clearly indicated the need to refine the

classical Kirchhoff-Love shell theories to better predict the stability and dynamic responses of composite cylindrical shell configurations. The Kirchhoff-Love theory assumes normals to the shell mid surface before deformation remain normal after deformation, effectively neglecting transverse shear strains. These classical theories predict shell panels that are too stiff, resulting in high frequencies and buckling loads. L.H. Donnell applied the Kirchhoff-Love theory to shallow cylindrical shell panels.

The need to include transverse shear effects was first recognized by Reissner (18), followed by Mindlin (12) who included rotary inertia effects in the dynamic analysis of plates. The Reissner-Mindlin theory assumes the cross section remains plane, but is allowed to rotate from the normal with respect to the mid surface after deformation. Extra independent degrees of freedom are included, which enables the transverse shear to be fully described by the shell mid surface degrees of freedom and the thickness coordinate. This first order theory does not satisfy the boundary conditions of zero transverse shear on the top and bottom surfaces of the laminate because of the constant shear angle assumed. The introduction of a correction factor helps to alleviate this problem.

Reddy (15), (17) and Soldatos (21) have recently applied a so called parabolic through the thickness shear strain distribution to analyze laminated anisotropic plates and shells. The in-plane displacements are cubic functions of the thickness

coordinate, satisfying zero transverse shear strain boundary conditions on the top and bottom surfaces of the laminate. The same independent degrees of freedom as used in Reissner-Mindlin theory are used here, but the need for a correction factor is eliminated.

It is this higher order transverse shear theory upon which the strain-displacement relations for this thesis are based.

OBJECTIVES

There are four main objectives to this thesis. First is the development of a higher order set of linear strain displacement relations for the cylindrical panel that incorporate parabolic transverse shear. The relations could be regarded as a linearized form of Sanders equations, applicable to deep panels (almost complete cylinders). The theory is not limited to shallow panels as is Donnell theory. (1) The strain displacement relations result in higher order constitutive relations for the panel. The second objective is the analytical solution for the fundamental frequencies and critical buckling loads of the cylindrical panel for different geometries and boundary conditions. Third, the method will be used to analyze the effects of shear deformation, rotary inertia, and radius of curvature. Intrinsic in this analysis is the determination of the maximum thickness to radius ratio allowed under the conditions of assuming zero transverse normal stress. And

fourthly, verification of the results by comparison with other approximate methods and classical methods, where applicable.

APPROACH

A logical approach is taken to complete this thesis. The displacement field for the anisotropic circular cylindrical panel, that is a function of the mid surface degrees of freedom, is developed based upon Reddy's (15), (16) ,(17) and Soldatos' (21) parabolic transverse shear strain model. Linear orthogonal curvilinear coordinates from Saada (19) are used to derive the strain displacement relations. These relations include higher order terms, applicable to the analysis of deep panels. Basic principles from Jones (9) are used to develop the higher order constitutive relations for the laminate. The kinetic energy, strain energy, and potential energy are each separately derived using principles from (11), (20), (5), and (7). Hamilton's principle is applied to extract the equations of motion and boundary conditions, which are then solved using the Galerkin technique.

To solve the equations, the degrees of freedom are approximated by admissible functions: those that satisfy geometric boundary conditions. The Galerkin equations are generated with the aid of MACSYMA (25) by substituting the admissible functions into the equations of motion and boundary conditions. A Fortran program is written which formulates the

eigenvalue problem from the Galerkin equations. The program yields the desired natural frequencies and/or buckling loads and their corresponding eigenvectors for a particular input geometry, ply layup, and boundary condition. Simply supported and clamped boundaries are analyzed.

Results are compared with other approximate solutions and classical solutions, where available. Also, to ensure valid results, the Galerkin technique is tested for convergence, and transverse normal stress effects are analyzed.

II. THEORETICAL DEVELOPMENT

The first step in the theoretical development for this thesis is the derivation of the displacement field based upon a through the thickness parabolic transverse shear strain distribution of the laminate. The LINEAR orthogonal curvilinear coordinate strain-displacement relations will then be derived. Next, anisotropic thick cylindrical shell panel theory will be discussed. From there, the kinetic energy, strain energy, and potential energy due to external forces will be derived and used in Hamilton's principle to formulate the equations of motion and boundary conditions for the panel. Finally, Galerkin's technique will be applied to approximate the differential equations of motion and boundary conditions. Galerkin's technique will be used for two different boundary conditions: all sides simply supported and all sides clamped.

STRAIN-DISPLACEMENT RELATIONS

The coordinate system for the circular cylindrical shell panel and the degrees of freedom to be used in the theory are shown in Figure 2.1. The x and y axes are located at the mid surface of the laminate ($z = 0$). The degrees of freedom $u_o(x,y,t)$, $v_o(x,y,t)$, and $w(x,y,t)$ are the laminate mid surface displacements in the x, y, and z directions, respectively. The degrees of freedom $\psi_x(x,y,t)$ and $\psi_y(x,y,t)$ are the rotations of the laminate cross section from the normal at the mid surface

with respect to the x and y axes, respectively. R is the radius of curvature, h the laminate thickness, a the length in the x direction, and b the length in the y direction.

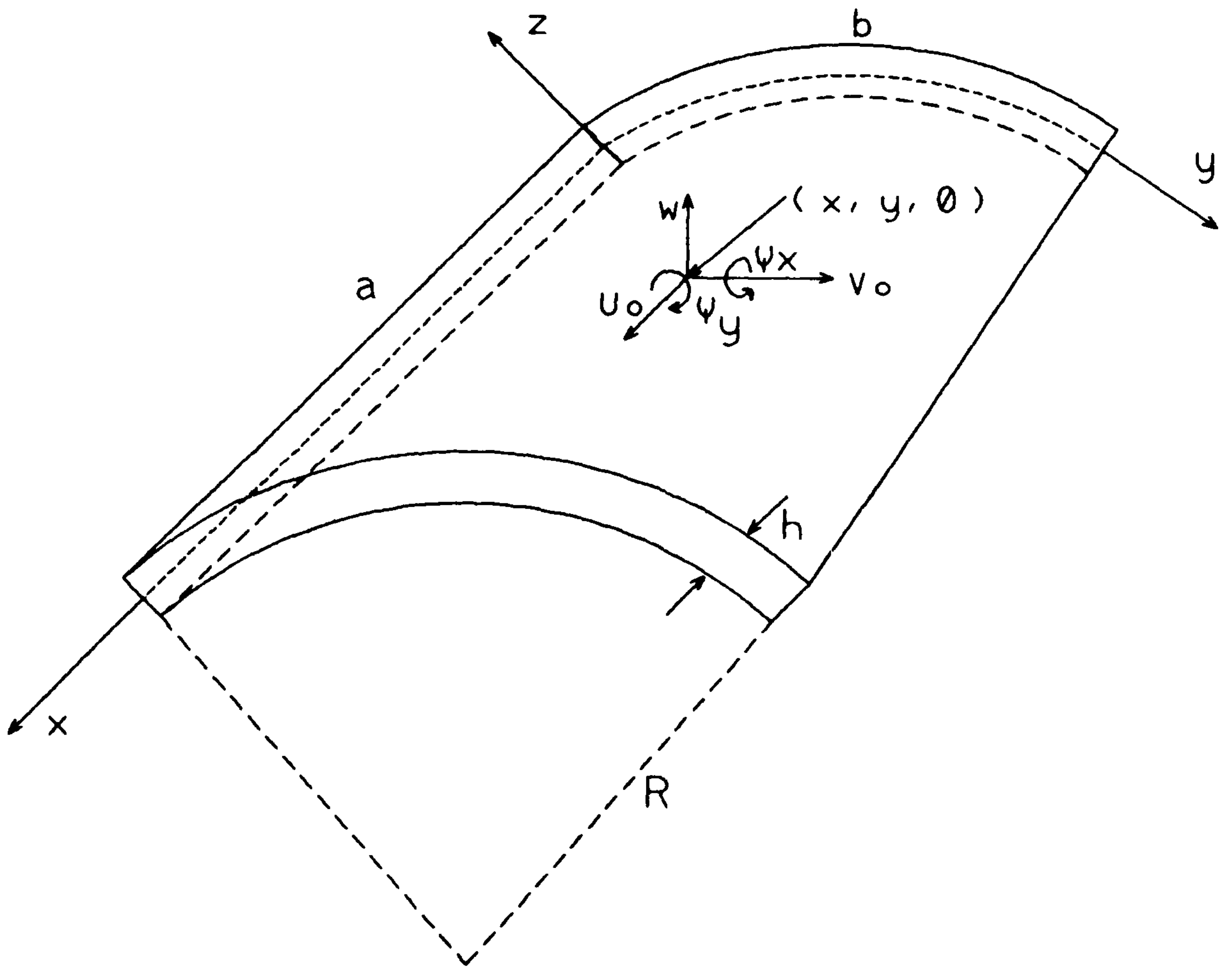

Figure 2.1 Shell panel coordinates and degrees of freedom

In order to determine the displacement field, the transverse shear strains, γ_{xz} and γ_{yz}, need to be modeled. In

classical laminated shell theory, through the thickness shear deformation is neglected according to the Kirchhoff Love hypothesis that plane cross sections remain plane and perpendicular to the laminate mid surface after deformation. A displacement field that is a first order function of z is required in classical shell theory. Bowlus (3),(4) and Palardy (13) in their flat plate work modeled transverse shear strain using Mindlin plate theory, which also required the use of a first order displacement field. Mindlin plate theory assumes the cross section remains plane, but is allowed to rotate from the normal with respect to the mid surface after deformation. The assumption of no cross sectional warping introduces error, especially at the top and bottom surfaces of the laminate, since the model does not match the boundary conditions of zero transverse shear strain there. This error is reduced by the introduction of a shear correction factor. This thesis models transverse shear strain parabolically wherein the strains are maximum at the laminate mid surface and are zero at the top and bottom surfaces, satisfying the boundary conditions. Figure 2.2 illustrates the transverse shear concepts discussed above.

To achieve the desired parabolic transverse shear, a higher order displacement field is required, as apposed to the first order displacement field used in the Classical and Mindlin cases. The coordinate displacements in the x and y directions, u and v, will be cubic functions of z; the displacement in the z direction, w, will be constant with respect to z. From Reddy (16) and Saada (19), the generalized displacement field is:

$$u(x,y,z,t) = u_o + z\psi_x + z^2\phi_1 + z^3\theta_1$$

$$v(x,y,z,t) = \left(1 + \frac{z}{R}\right)v_o + z\psi_y + z^2\phi_2 + z^3\theta_2$$

$$w(x,y,t) = w \tag{2.1}$$

where ϕ_1, ϕ_2, θ_1, and θ_2 will be chosen to satisfy the boundary conditions of zero transverse shear strain at the laminate top

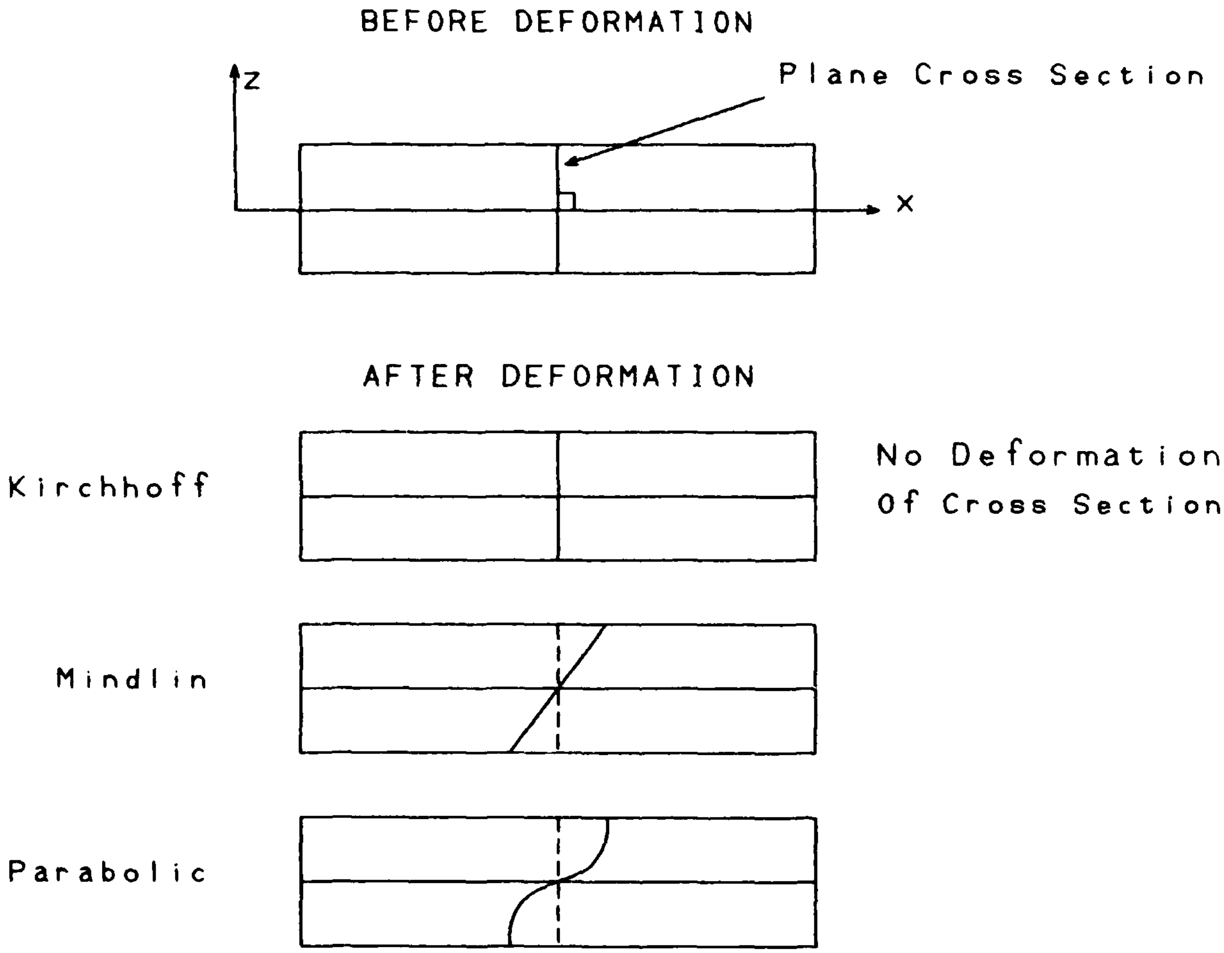

Figure 2.2 Transverse Shear Strain Models

and bottom surfaces.

Linear orthogonal curvilinear coordinates are used to develop the strain-displacement relations. (14),(19) For a circular cylindrical shell panel these relations reduce to:

$$\varepsilon_x = u,_x$$
$$\varepsilon_y = \frac{1}{1 + \frac{z}{R}} \left[v,_y + \frac{w}{R}\right]$$
$$\gamma_{xy} = \frac{1}{1 + \frac{z}{R}} u,_y + v,_x$$
$$\gamma_{yz} = \frac{1}{1 + \frac{z}{R}} \left[w,_y - \frac{v}{R}\right] + v,_z$$
$$\gamma_{xz} = u,_z + w,_x \tag{2.2}$$

where ( ),$_x$ represents partial differentiation with respect to x and so on. ε_z is assumed equal to zero. This implies that a change in length in the normal (z) direction of a cross section perpendicular to the mid surface is not considered, and is regarded as an accepted inconsistency in plate and shell theory. In reality, ε_z is not zero, but is small compared to the other strains. For the laminate, it means there are discontinuities in ε_z at the lamina boundaries, but they too are small.

The Donnell cylindrical shell panel equations assume $\frac{z}{R} \approx 0$ in Eq (2.2). As shown in Eq (2.33a) later in this chapter, this assumption limits Donnell theory to be valid only for small $\frac{h}{R}$ ratios. With no transverse shear, the maximum h/R limit under Donnell assumptions is approximately 1/500. (23) As will be shown, with transverse shear included, the Donnell equations are valid up to h/R equal to approximately 1/50. (16)

For simplicity this thesis assumes $\frac{z}{R} \cong 0$ for the transverse shear strains, γ_{yz} and γ_{xz}, only. (The limitations of the model resulting from these simplifications are discussed in Appendix A.) For the membrane strains ε_x, ε_y, and γ_{xy}, the following polynomial expansion is made:

$$\frac{1}{1 + \frac{z}{R}} \cong 1 - \frac{z}{R}$$

This approximation allows the strain-displacement relations to be valid for deep panels, with an $\frac{h}{R}$ maximum limit of approximately 1/5. (See Dennis (8) and Appendix A.) The transverse shear strains in Eq (2.2) become:

$$\gamma_{yz} = v,_z + w,_y - \frac{v}{R}$$
$$\gamma_{xz} = u,_z + w,_x$$

If one sets $\gamma_{yz}(x,y,\pm h/2,t) = 0$ and $\gamma_{xz}(x,y,\pm h/2,t) = 0$ to satisfy the laminate surface boundary conditions, then from Eq (2.1) it can be shown that (see Appendix A):

$$\phi_1 = \phi_2 = 0$$

$$\theta_1 = k(\psi_x + w,_x) \quad , \quad \theta_2 = k(\psi_y + w,_y)$$

where $k = -\frac{4}{3h^2}$. The displacement field then becomes:

$$u(x,y,z,t) = u_o + z\psi_x + z^3 k(\psi_x + w,_x)$$
$$v(x,y,z,t) = \left(1 + \frac{z}{R}\right)v_o + z\psi_y + z^3 k(\psi_y + w,_y)$$
$$w(x,y,z,t) = w \qquad (2.3)$$

Using this displacement field in Eq (2.2), the

strain-displacement relations become

$$\varepsilon_x = u_{o,x} + z\psi_{x,x} + z^3 k(\psi_{x,x} + w_{,xx})$$

$$\varepsilon_y = v_{o,y} + \frac{w}{R} + z\psi_{y,y} - \frac{1}{R} z^2 \psi_{y,y} + z^3 k(\psi_{y,y} + w_{,yy}) - \frac{1}{R} z^4 k(\psi_{y,y} + w_{,yy})$$

$$\gamma_{xy} = u_{o,y} + v_{o,x} + z\left[\psi_{x,y} + \psi_{y,x} + \frac{1}{2R}(v_{o,x} - u_{o,y})\right] - \frac{1}{R} z^2 \psi_{x,y} + z^3 k(\psi_{x,y} + \psi_{y,x} + 2w_{,xy}) - \frac{1}{R} z^4 k(\psi_{x,y} + w_{,xy})$$

$$\gamma_{yz} = \psi_y + w_{,y} + 3kz^2(\psi_y + w_{,y})$$

$$\gamma_{xz} = \psi_x + w_{,x} + 3kz^2(\psi_x + w_{,x}) \qquad (2.4)$$

Shorthand notation can be introduced to rewrite the strains as follows:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \gamma_{yz}^o \\ \gamma_{xz}^o \end{Bmatrix} + z \begin{Bmatrix} \kappa_x^o \\ \kappa_y^o \\ \kappa_{xy}^o \\ 0 \\ 0 \end{Bmatrix} + z^2 \begin{Bmatrix} 0 \\ \kappa_y^1 \\ \kappa_{xy}^1 \\ \kappa_{yz}^1 \\ \kappa_{xz}^1 \end{Bmatrix} + z^3 \begin{Bmatrix} \kappa_x^2 \\ \kappa_y^2 \\ \kappa_{xy}^2 \\ 0 \\ 0 \end{Bmatrix} + z^4 \begin{Bmatrix} 0 \\ \kappa_y^3 \\ \kappa_{xy}^3 \\ 0 \\ 0 \end{Bmatrix} \qquad (2.5)$$

(Note the superscripts on the κ terms are not exponents. They are for identification purposes only and simply distinguish

among the high and low order curvature terms.) The strains at the laminate mid surface are:

$$\begin{Bmatrix} \varepsilon_x^{\,o} \\ \varepsilon_y^{\,o} \\ \gamma_{xy}^{\,o} \\ \gamma_{yz}^{\,o} \\ \gamma_{xz}^{\,o} \end{Bmatrix} = \begin{Bmatrix} u_{o,x} \\ v_{o,y} + \frac{w}{R} \\ u_{o,y} + v_{o,x} \\ \psi_y + w,_y \\ \psi_x + w,_x \end{Bmatrix} \tag{2.6}$$

and the curvature terms ($\varkappa$) due to bending and shear deformation are defined as follows:

$$\begin{Bmatrix} \varkappa_x^{\,o} \\ \varkappa_y^{\,o} \\ \varkappa_{xy}^{\,o} \end{Bmatrix} = \begin{Bmatrix} \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} + \frac{1}{2R}\,(v_{o,x} - u_{o,y}) \end{Bmatrix}$$

$$\begin{Bmatrix} \varkappa_y^{\,1} \\ \varkappa_{xy}^{\,1} \\ \varkappa_{yz}^{\,1} \\ \varkappa_{xz}^{\,1} \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{R}\,\psi_{y,y} \\ -\frac{1}{R}\,\psi_{x,y} \\ 3k(\psi_y + w,_y) \\ 3k(\psi_x + w,_x) \end{Bmatrix}$$

$$\begin{Bmatrix} \varkappa_x^{\,2} \\ \varkappa_y^{\,2} \\ \varkappa_{xy}^{\,2} \end{Bmatrix} = \begin{Bmatrix} k(\psi_{x,x} + w,_{xx}) \\ k(\psi_{y,y} + w,_{yy}) \\ k(\psi_{x,y} + \psi_{y,x} + 2w,_{xy}) \end{Bmatrix}$$

$$\begin{Bmatrix} \varkappa_y^{\,3} \\ \varkappa_{xy}^{\,3} \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{R}\,k(\psi_{y,y} + w,_{yy}) \\ -\frac{1}{R}\,k(\psi_{x,y} + w,_{xy}) \end{Bmatrix} \tag{2.7}$$

ANISOTROPIC THICK CYLINDRICAL SHELL PANEL THEORY

Lamination theory incorporates constitutive relationships for an orthotropic lamina through the shell panel thickness resulting in expressions which approximate force resultants in terms of displacement functions. This theory provides concepts which are required in the subsequent development of the equations of motion and boundary conditions. The constitutive relationships are developed for the basic building block, the lamina, to the end result, the structural laminate. The end results of this section are the laminate stiffness terms and force resultants.

The plane stress constitutive relationships for a single orthotropic layer in the principle coordinate system shown in Figure 2.3 are

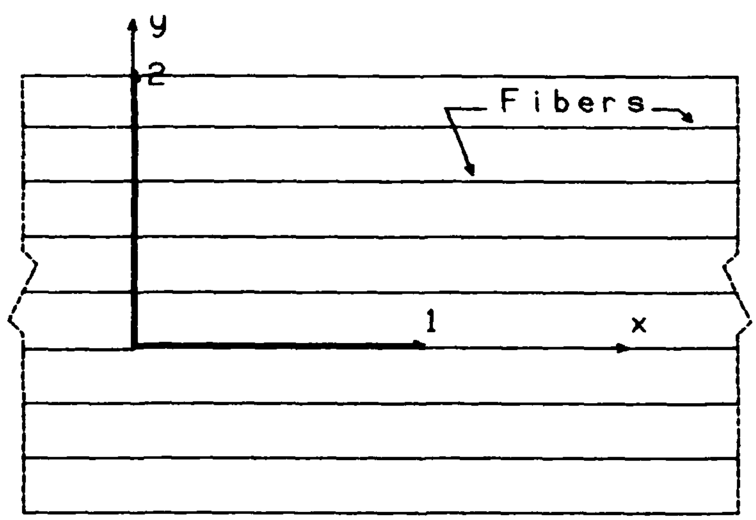

Figure 2.3 Lamina Material Coordinates

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}
$$

$$
\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} S_{44} & 0 \\ 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} \tag{2.8}
$$

Note that $\sigma_z = 0$ for plane stress. That is, the individual laminae are considered to be thin enough that the average value of σ_z across the thickness is negligible. The S_{ij} are compliance terms and may be written in terms of the lamina engineering constants as:

$$
\begin{aligned}
S_{11} &= \frac{1}{E_1} \\
S_{12} &= -\frac{\nu_{21}}{E_2} \\
S_{22} &= \frac{1}{E_2} \\
S_{66} &= \frac{1}{G_{12}} \\
S_{44} &= \frac{1}{G_{23}} \\
S_{55} &= \frac{1}{G_{13}}
\end{aligned} \tag{2.9}
$$

where E_i are Young's moduli in the ith direction, ν_{ij} is Poisson's ratio for transverse strain in the jth direction when stressed in the ith direction, and G_{ij} is the shear modulus in the i-j plane.

Equation (2.8) may be inverted to give the relationship of the stresses in terms of the strains:

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}
$$

$$
\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{2.10}
$$

where Q_{ij} are the reduced stiffness terms and are defined as:

$$
\begin{aligned}
Q_{11} &= \frac{E_1}{1 - \nu_{12}\nu_{21}} \\
Q_{12} &= \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} \\
Q_{22} &= \frac{E_2}{1 - \nu_{12}\nu_{21}} \\
Q_{66} &= G_{12} \\
Q_{44} &= G_{23} \\
Q_{55} &= G_{31}
\end{aligned} \tag{2.11}
$$

A structural laminate consists of N laminae oriented at different angles with respect to each other. The previous constitutive relations apply only to Figure 2.3 where the lamina-fixed 1-2 axis system is aligned with the laminate (or global) x-y axis system. If the 1-2 axis system is not aligned with the x-y axis system but rather is at an angle θ (see Figure 2.4), the reduced stiffness matrix, $[Q_{ij}]$, must be transformed.

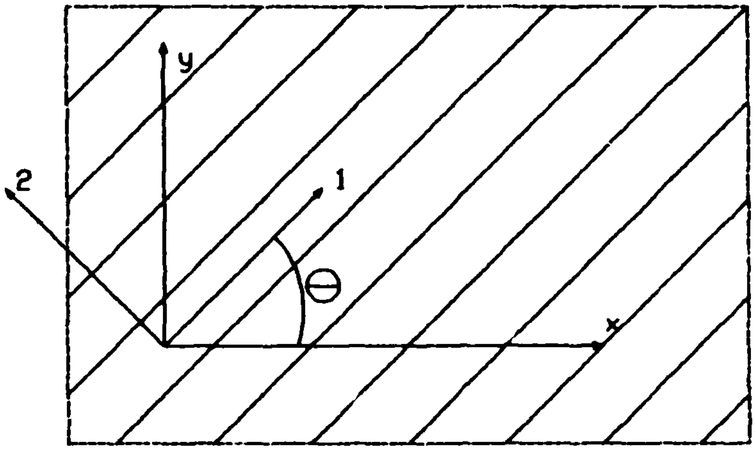

Figure 2.4 Arbitrary Lamina Coordinates

The transformation matrices applied to the stiffness terms in Eq (2.10) to reflect the shift in the laminae axes are defined below:

$$\text{For} \quad \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}, \quad T = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix},$$

$$\text{For} \quad \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix}, \quad T = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$.

The transformed reduced stiffness matrices then become:

$$\left[\bar{Q}_{ij} \right] = \left[T \right] \left[Q_{ij} \right] \left[T \right]^T$$

From (9), the lamina constitutive relationships can now be expressed in laminate coordinates as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}_k \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{2.12}$$

where k denotes the k_{th} lamina and the individual $\bar{Q}_{ij}$ are computed as:

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta)$$

$$\bar{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^3\theta$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta)$$

$$\bar{Q}_{44} = Q_{44}\cos^2\theta + Q_{55}\sin^2\theta$$

$$\bar{Q}_{45} = (Q_{44} - Q_{55})\cos\theta\sin\theta$$

$$\bar{Q}_{55} = Q_{55}\cos^2\theta + Q_{44}\sin^2\theta \tag{2.13}$$

Finally, substituting the expressions for the strains in Eq (2.5) into the constitutive relations in Eq (2.12), the stress in the k_{th} lamina of the structural laminate is expressed as:

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left[\begin{Bmatrix} \varepsilon_x^{\,o} \\ \varepsilon_y^{\,o} \\ \gamma_{xy}^{\,o} \end{Bmatrix} + z \begin{Bmatrix} \kappa_x^{\,o} \\ \kappa_y^{\,o} \\ \kappa_{xy}^{\,o} \end{Bmatrix} + z^2 \begin{Bmatrix} 0 \\ \kappa_y^{\,1} \\ \kappa_{xy}^{\,1} \end{Bmatrix} \right.
$$

$$
\left. + z^3 \begin{Bmatrix} \kappa_x^{\,2} \\ \kappa_y^{\,2} \\ \kappa_{xy}^{\,2} \end{Bmatrix} + z^4 \begin{Bmatrix} 0 \\ \kappa_y^{\,3} \\ \kappa_{xy}^{\,3} \end{Bmatrix} \right]
$$

$$
\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}_k \left[\begin{Bmatrix} \gamma_{yz}^{\,o} \\ \gamma_{xz}^{\,o} \end{Bmatrix} + z^2 \begin{Bmatrix} \kappa_{yz}^{\,1} \\ \kappa_{xz}^{\,1} \end{Bmatrix} \right] \qquad (2.14)
$$

The resultant forces and moments and the higher order resultant quantities acting on the laminate are obtained by integrating the stresses in each lamina through the laminate thickness. Thus, for the laminate with N laminae shown in Figure 2.5

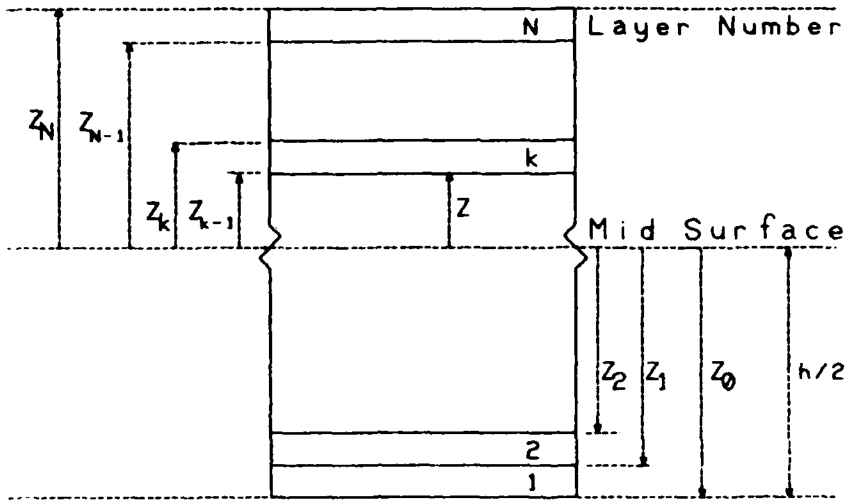

Figure 2.5 Geometry of an N layered laminate

the resultant forces and moments and higher order quantities are:

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix}, \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix}, \begin{Bmatrix} S_1 \\ S_2 \\ S_6 \end{Bmatrix}, \begin{Bmatrix} P_1 \\ P_2 \\ P_6 \end{Bmatrix}, \begin{Bmatrix} L_1 \\ L_2 \\ L_6 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} (1, z, z^2, z^3, z^4) dz$$

$$= \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k (1, z, z^2, z^3, z^4) dz$$

$$\begin{Bmatrix} Q_2 \\ Q_1 \end{Bmatrix}, \begin{Bmatrix} R_2 \\ R_1 \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} (1, z^2) dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}_k (1, z^2) dz \tag{2.15}$$

where $\{N_i\}$ and $\{Q_i\}$ are the resultant forces, $\{M_i\}$ are the resultant moments, and $\{S_i\}$, $\{P_i\}$, $\{L_i\}$, and $\{R_i\}$ are the higher order quantities resulting from the higher order strain-displacement relations.

By substituting Eq (2.14) into Eq (2.15), thereby expressing the stresses in terms of the mid surface displacement quantities and the transformed reduced stiffness matrices, the integration is simplified because the mid surface values are independent of z and can come out of the integral and summation signs. (9) This allows the following notation to be adopted for the integrated laminate stiffness matrices:

$$(A_{ij},\ B_{ij},\ D_{ij},\ E_{ij},\ F_{ij},\ G_{ij},\ H_{ij},\ I_{ij},\ J_{ij}) =$$

$$\sum_{k=1}^{N} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{z_{k-1}}^{z_k} (1, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8)\,dz \qquad i,j = 1,2,6$$

For the transverse shear:

$$(A_{ij},\ D_{ij},\ F_{ij}) = \sum_{k=1}^{N} \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}_k \int_{z_{k-1}}^{z_k} (1, z^2, z^4)\,dz \qquad i,j = 4,5$$

$$(2.16)$$

Now Eq (2.15) may be written as:

$$
\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix}
\begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix}
\begin{Bmatrix} S_1 \\ S_2 \\ S_6 \end{Bmatrix}
\begin{Bmatrix} P_1 \\ P_2 \\ P_6 \end{Bmatrix}
\begin{Bmatrix} L_1 \\ L_2 \\ L_6 \end{Bmatrix}
=
\begin{bmatrix}
[A_{ij}] & [B_{ij}] & [D_{ij}] & [E_{ij}] & [F_{ij}] \\
[B_{ij}] & [D_{ij}] & [E_{ij}] & [F_{ij}] & [G_{ij}] \\
[D_{ij}] & [E_{ij}] & [F_{ij}] & [G_{ij}] & [H_{ij}] \\
[E_{ij}] & [F_{ij}] & [G_{ij}] & [H_{ij}] & [I_{ij}] \\
[F_{ij}] & [G_{ij}] & [H_{ij}] & [I_{ij}] & [J_{ij}]
\end{bmatrix}
\begin{Bmatrix} \varepsilon_x^{\,0} \\ \varepsilon_y^{\,0} \\ \gamma_{xy}^{\,0} \end{Bmatrix}
\begin{Bmatrix} \kappa_x^{\,0} \\ \kappa_y^{\,0} \\ \kappa_{xy}^{\,0} \end{Bmatrix}
\begin{Bmatrix} 0 \\ \kappa_y^{\,1} \\ \kappa_{xy}^{\,1} \end{Bmatrix}
\begin{Bmatrix} \kappa_x^{\,2} \\ \kappa_y^{\,2} \\ \kappa_{xy}^{\,2} \end{Bmatrix}
\begin{Bmatrix} 0 \\ \kappa_y^{\,3} \\ \kappa_{xy}^{\,3} \end{Bmatrix}
$$

$$
i, j = 1, 2, 6
$$

$$
\begin{Bmatrix} Q_2 \\ Q_1 \end{Bmatrix}
\begin{Bmatrix} R_2 \\ R_1 \end{Bmatrix}
=
\begin{bmatrix}
A_{44} & A_{45} & D_{44} & D_{45} \\
A_{45} & A_{55} & D_{45} & D_{55} \\
D_{44} & D_{45} & F_{44} & F_{45} \\
D_{45} & D_{55} & F_{45} & F_{55}
\end{bmatrix}
\begin{Bmatrix} \gamma_{yz}^{\,0} \\ \gamma_{xz}^{\,0} \end{Bmatrix}
\begin{Bmatrix} \kappa_{yz}^{\,1} \\ \kappa_{xz}^{\,1} \end{Bmatrix}
\tag{2.17}
$$

where the large matrix above is (15 x 15) and each of its submatrices are the (3 x 3) matrices in Eq (2.16).

EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

The displacement field, strain displacement relations, and the laminate resultant quantities in Eq (2.17) will now be used in the energy formulation to find the equations of motion and boundary conditions.

The fundamental equation used in this development is Hamilton's Principle:

$$\int_{t_1}^{t_2} \left[\delta T - \delta U - \delta V \right] dt = 0 \tag{2.18}$$

where

T = kinetic energy

U = strain energy

V = potential energy due to external forces, and δ is the first variation. This section will be devoted to the derivation of the kinetic energy, strain energy, and potential energy, and finally to the application of Hamilton's principle. The result will be five coupled partial differential equations of motion plus their associated boundary conditions.

The kinetic energy for the structure is defined as

$$T = \int_0^b \int_0^a \int_{-h/2}^{h/2} \frac{1}{2}\rho \left[\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right] dz dx dy \tag{2.19}$$

where ρ is the mass density (11), (20).

Taking the partial time derivatives of the midplane

displacements and squaring, the following is obtained:

$$\dot{u}^2 = \dot{u}_o^{\,2} + (2z + 2kz^3)\dot{u}_o\dot{\psi}_x + 2kz^3\dot{u}_o\dot{w},_x + (z^2 + 2kz^4 + k^2z^6)\dot{\psi}_x^{\,2} + (2kz^4 + 2k^2z^6)\dot{\psi}_x\dot{w},_x + k^2z^6\dot{w},_x^{\,2}$$

$$\dot{v}^2 = \left[1 + 2\frac{z}{R}\right]\dot{v}_o^{\,2} + \left[1 + \frac{z}{R}\right](2z + 2kz^3)\dot{v}_o\dot{\psi}_y + 2\left[1 + \frac{z}{R}\right]kz^3\dot{v}_o\dot{w},_y + (z^2 + 2kz^4 + k^2z^6)\dot{\psi}_y^{\,2} + (2kz^4 + 2k^2z^6)\dot{\psi}_y\dot{w},_y + k^2z^6\dot{w},_y^{\,2}$$

$$\dot{w}^2 = \dot{w}^2 \tag{2.20}$$

By substituting Eq (2.20) into Eq (2.19) and making the following definitions for the mass moments of inertia:

$$(I_1, I_2, I_3, I_4, I_5, I_7) = \int_{-h/2}^{h/2} \rho(1, z, z^2, z^3, z^4, z^6)dz$$

$$\bar{I}_1' = I_1 + \frac{2}{R}I_2$$

$$\bar{I}_2 = I_2 + kI_4$$

$$\bar{I}_2' = I_2 + \frac{1}{R}I_3 + kI_4 + \frac{k}{R}I_5$$

$$\bar{I}_3 = -kI_4$$

$$\bar{I}_3' = -kI_4 - \frac{k}{R}I_5$$

$$\bar{I}_4 = I_3 + 2kI_5 + k^2I_7$$

$$\bar{I}_5 = -kI_5 - k^2I_7 \tag{2.21}$$

Eq (2.19) becomes:

$$T = \frac{1}{2}\int_0^b \int_0^a \Big[I_1\dot{u}_o^{\,2} + 2\bar{I}_2\dot{u}_o\dot{\psi}_x - 2\bar{I}_3\dot{u}_o\dot{w},_x + \bar{I}_4\dot{\psi}_x^{\,2} - 2\bar{I}_5\dot{\psi}_x\dot{w},_x$$

$$+ k^2 I_7\dot{w},_x^{\,2} + \bar{I}_1{}'\dot{v}_o^{\,2} + 2\bar{I}_2{}'\dot{v}_o\dot{\psi}_y - 2\bar{I}_3{}'\dot{v}_o\dot{w},_y$$

$$+ \bar{I}_4\dot{\psi}_y^{\,2} - 2\bar{I}_5\dot{\psi}_y\dot{w},_y + k^2 I_7\dot{w},_y^{\,2} + I_1\dot{w}^2\Big]dxdy \quad (2.22)$$

Taking the first variation and collecting terms gives the following result:

$$\delta T = \int_0^b \int_0^a \Big[(I_1\dot{u}_o + \bar{I}_2\dot{\psi}_x - \bar{I}_3\dot{w},_x)\delta\dot{u}_o + (\bar{I}_2\dot{u}_o + \bar{I}_4\dot{\psi}_x - \bar{I}_5\dot{w},_x)\delta\dot{\psi}_x$$

$$+ (-\bar{I}_3\dot{u}_o - \bar{I}_5\dot{\psi}_x + k^2 I_7\dot{w},_x)\delta\dot{w},_x + (\bar{I}_1{}'\dot{v}_o + \bar{I}_2{}'\dot{\psi}_y - \bar{I}_3{}'\dot{w},_y)\delta\dot{v}_o$$

$$+ (\bar{I}_2{}'\dot{v}_o + \bar{I}_4\dot{\psi}_y - \bar{I}_5\dot{w},_y)\delta\dot{\psi}_y + (-\bar{I}_3{}'\dot{v}_o - \bar{I}_5\dot{\psi}_y + k^2 I_7\dot{w},_y)\delta\dot{w},_y$$

$$+ I_1\dot{w}\delta\dot{w}\Big]dxdy \quad (2.23)$$

The following steps are taken to obtain the final form of the variation in kinetic energy: (1) Integrate Eq (2.23) by parts with respect to x and y (for this procedure, see Appendix B) for the terms $\delta\dot{w},_x$ and $\delta\dot{w},_y$; (2) Integrate the complete resulting expression by parts with respect to time; (3) Collect terms; (4) Neglect the variations of the degrees of freedom at the endpoints t_1 and t_2; and (5) Neglect time dependant boundaries, since only harmonic problems are considered. The following is obtained:

$$\int_{t_1}^{t_2} \delta T dt = \int_{t_1}^{t_2} \int_0^b \int_0^a \Big\{ - \left(I_1\ddot{u}_o + \bar{I}_2\ddot{\psi}_x - \bar{I}_3\ddot{w}_{,x}\right)\delta u_o - \left(\bar{I}_1{}'\ddot{v}_o + \bar{I}_2{}'\ddot{\psi}_y - \bar{I}_3{}'\ddot{w}_{,y}\right)\delta v_o - \left(\bar{I}_3\ddot{u}_{o,x} + \bar{I}_5\ddot{\psi}_{x,x} + \bar{I}_3{}'\ddot{v}_{o,y} - k^2 I_7(\ddot{w}_{,xx} + \ddot{w}_{,yy}) + \bar{I}_5\ddot{\psi}_{y,y} + I_1\ddot{w}\right)\delta w - \left(\bar{I}_2\ddot{u}_o + \bar{I}_4\ddot{\psi}_x - \bar{I}_5\ddot{w}_{,x}\right)\delta\psi_x - \left(\bar{I}_2{}'\ddot{v}_o + \bar{I}_4\ddot{\psi}_y - \bar{I}_5\ddot{w}_{,y}\right)\delta\psi_y \Big\} dxdydt \tag{2.24}$$

The strain energy is developed following the procedures outlined in (11), (14) and (20). The first variation of the strain energy may be written:

$$\delta U = \int_0^b \int_0^a \int_{-h/2}^{h/2} \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz}\right) dzdxdy \tag{2.25}$$

By substituting the strain-displacement relations in Eq (2.5) into Eq (2.25), integrating with respect to z, and using the resultant quantities in Eq (2.15), the first variation of strain energy may be rewritten as:

$$\delta U = \int_0^b \int_0^a \Big[N_1 \delta \varepsilon_x{}^{o} + M_1 \delta \kappa_x{}^{o} + P_1 \delta \kappa_x{}^{2} + N_2 \delta \varepsilon_y{}^{o} + M_2 \delta \kappa_y{}^{o} + S_2 \delta \kappa_y{}^{1}$$

$$+ P_2 \delta \kappa_y{}^{2} + L_2 \delta \kappa_y{}^{3} + N_6 \delta \gamma_{xy}{}^{o} + M_6 \delta \kappa_{xy}{}^{o} + S_6 \delta \kappa_{xy}{}^{1} + P_6 \delta \kappa_{xy}{}^{2}$$

$$+ L_6 \delta \kappa_{xy}{}^{3} + Q_2 \delta \gamma_{yz}{}^{o} + R_2 \delta \kappa_{yz}{}^{1} + Q_1 \delta \gamma_{xz}{}^{o} + R_1 \delta \kappa_{xz}{}^{1} \Big] dxdy \tag{2.26}$$

Substituting the expressions for the mid surface strains and curvatures in Eqs (2.6) and (2.7) into Eq (2.26) and then collecting terms, the following is obtained for the strain energy:

$$\delta U = \int_0^b \int_0^a \Big\{ N_1 \delta u_{o,x} + \Big[N_6 - \frac{1}{2R} M_6 \Big] \delta u_{o,y} + N_2 \delta v_{o,y} + \frac{1}{R} N_2 \delta w +$$

$$\Big[N_6 + \frac{1}{2R} M_6 \Big] \delta v_{o,x} + kP_1 \delta w,_{xx} + \Big[kP_2 - k\frac{1}{R} L_2 \Big] \delta w,_{yy} +$$

$$\Big[2kP_6 - k\frac{1}{R} L_6 \Big] \delta w,_{xy} + (Q_2 + 3kR_2) \delta w,_{y} + (Q_1 + 3kR_1) \delta w,_{x} +$$

$$(M_1 + kP_1) \delta \psi_{x,x} + (Q_1 + 3kR_1) \delta \psi_x + (Q_2 + 3kR_2) \delta \psi_y +$$

$$\Big[M_2 - \frac{1}{R} S_2 + kP_2 - \frac{1}{R} kL_2 \Big] \delta \psi_{y,y} + (M_6 + kP_6) \delta \psi_{y,x} +$$

$$\Big[M_6 - \frac{1}{R} S_6 + kP_6 - \frac{1}{R} kL_6 \Big] \delta \psi_{x,y} \Big\} dxdy \tag{2.27}$$

Eq (2.27) is integrated by parts according to Appendix B to obtain:

$$
\begin{aligned}
\delta U = \int_0^b \int_0^a \Big\{ & \Big[- N_{1,x} - N_{6,y} + \frac{1}{2R} M_{6,y}\Big]\delta u_o + \Big[- N_{2,y} - N_{6,x} - \\
& \frac{1}{2R} M_{6,x}\Big]\delta v_o + \Big[k(P_{1,xx} + P_{2,yy} + 2P_{6,xy}) - Q_{2,y} - Q_{1,x} - \\
& 3k(R_{2,y} + R_{1,x}) + \frac{1}{R}[N_2 - k(L_{2,yy} + L_{6,xy})]\Big]\delta w + \Big[3kR_1 - \\
& k(P_{1,x} + P_{6,y}) - M_{1,x} - M_{6,y} + Q_1 + \frac{1}{R}(S_{6,y} + kL_{6,y})\Big]\delta\psi_x + \\
& \Big[3kR_2 - k(P_{2,y} + P_{6,x}) - M_{2,y} + \frac{1}{R}(S_{2,y} + kL_{2,y}) - \\
& M_{6,x} + Q_2\Big]\delta\psi_y \Big\} dxdydt \\
+ \int_0^b \Big\{ & N_1 \delta u_o + \Big[N_6 + \frac{1}{2R} M_6\Big]\delta v_o + \Big[-k(P_{1,x} + 2P_{6,y}) + Q_1 + \\
& 3kR_1 + \frac{1}{R} kL_{6,y}\Big]\delta w + (M_1 + 2kP_1)\delta\psi_x + (M_6 + kP_6)\delta\psi_y \Big\}\Big|_{x=0}^{x=a} dy \\
+ \int_0^a \Big\{ & \Big[N_6 - \frac{1}{2R} M_6\Big]\delta u_o + N_2 \delta v_o + \Big[-k(P_{2,y} + 2P_{6,x}) + Q_2 + \\
& 3kR_2 + \frac{1}{R} k(L_{2,y} + L_{6,x})\Big]\delta w + \Big[M_6 + kP_6 + \frac{1}{R}(-S_6 - kL_6)\Big]\delta\psi_x \\
& + \Big[M_2 + 2kP_2 + \frac{1}{R}(-S_2 - 2kL_2)\Big]\delta\psi_y \Big\} \Big|_{y=0}^{y=b} dx \\
+ k & \Big[2P_6 - \frac{1}{R} L_6\Big]\delta w \,\Big|_{y=0}^{y=b} \Big|_{x=0}^{x=a}
\end{aligned}
\tag{2.28}
$$

The last component of the energy formulation to consider is

the potential energy due to external forces. The only external forces considered are inplane forces that will bifurcate the laminate in buckling: ie, those forces that will create out-of-plane displacement. In-plane inextensibility is assumed in the bifurcation analysis. Therefore, only nonlinear bending strains need to be considered. The linear and nonlinear bending portions of the membrane strains at the laminate mid surface are from Saada (19):

$$\varepsilon_x = \frac{1}{2} w,_x^{\,2}$$

$$\varepsilon_y = \frac{w}{R} + \frac{1}{2} w,_y^{\,2}$$

$$\gamma_{xy} = w,_x w,_y$$

The expression for the potential energy then becomes:

$$V = \int_0^b \int_0^a \left\{ \bar{N}_1 \frac{1}{2} w,_x^{\,2} + \bar{N}_2 \left[\frac{w}{R} + \frac{1}{2} w,_y^{\,2} \right] + \bar{N}_6 w,_x w,_y \right\} dxdy \tag{2.29}$$

where $\bar{N}_1$ and $\bar{N}_2$ are externally applied loads per length in the x and y directions, respectively, and $\bar{N}_6$ is the externally applied inplane shear load per length. (See (5) and (7).) This thesis will only be concerned with axial buckling in the x direction, but will develop a general formulation. Taking the first variation and collecting terms:

$$\delta V = \int_0^b \int_0^a \left\{ (\bar{N}_1 w,_x + \bar{N}_6 w,_y)\delta w,_x + (\bar{N}_2 w,_y + \bar{N}_6 w,_x)\delta w,_y + \frac{1}{R}\bar{N}_2 \delta w \right\} dxdy \tag{2.30}$$

After integrating by parts and collecting terms, the final form

of the potential energy becomes:

$$\delta V = \int_0^b \int_0^a \left\{ -\bar{N}_1 w,_{xx} - 2\bar{N}_6 w,_{xy} + \bar{N}_2 \left(\frac{1}{R} - w,_{yy} \right) \right\} \delta w dx dy$$

$$+ \int_0^b (\bar{N}_1 w,_x + \bar{N}_6 w,_y) \delta w \Big|_{x=0}^{x=a} dy + \int_0^a (\bar{N}_2 w,_y + \bar{N}_6 w,_x) \delta w \Big|_{y=0}^{y=b} dx \qquad (2.31)$$

The expressions for the first variations of kinetic, strain, and potential energy in Eqs (2.24), (2.28), and (2.31) are used in Hamilton's principle, Eq (2.18), to obtain:

$$\int_{t_1}^{t_2} \int_0^b \int_0^a \Big\{ \Big(-I_1 \ddot{u}_o - \bar{I}_2 \ddot{\psi}_x + \bar{I}_3 \ddot{w},_x + N_{1,x} + N_{6,y} - \frac{1}{2R} M_{6,y} \Big) \delta u_o$$

$$+ \Big(-\bar{I}_1{}' \ddot{v}_o - \bar{I}_2{}' \ddot{\psi}_y + \bar{I}_3{}' \ddot{w},_y + N_{2,y} + N_{6,x} + \frac{1}{2R} M_{6,x} \Big) \delta v_o$$

$$+ \Big[-\bar{I}_3 \ddot{u}_{o,x} - \bar{I}_5 \ddot{\psi}_{x,x} - \bar{I}_3{}' \ddot{v}_{o,y} + k^2 I_7 (\ddot{w},_{xx} + \ddot{w},_{yy}) -$$

$$\bar{I}_5 \ddot{\psi}_{y,y} - I_1 \ddot{w} - k(P_{1,xx} + P_{2,yy} + 2P_{6,xy}) + Q_{2,y} +$$

$$Q_{1,x} + 3k(R_{2,y} + R_{1,x}) - \frac{1}{R}[N_2 - k(L_{2,yy} + L_{6,xy})] +$$

$$\bar{N}_1 w,_{xx} + 2\bar{N}_6 w,_{xy} - \bar{N}_2 \Big(\frac{1}{R} - w,_{yy} \Big) \Big] \delta w$$

$$+ \Big(-\bar{I}_2 \ddot{u}_o - \bar{I}_4 \ddot{\psi}_x + \bar{I}_5 \ddot{w},_x + k(P_{1,x} + P_{6,y}) + M_{1,x} + M_{6,y} -$$

$$3kR_1 - Q_1 - \frac{1}{R}(S_{6,y} + kL_{6,y}) \Big) \delta \psi_x$$

$$+ \Big(-\bar{I}_2{}' \ddot{v}_o - \bar{I}_4 \ddot{\psi}_y + \bar{I}_5 \ddot{w},_y + k(P_{2,y} + P_{6,x}) + M_{2,y} + M_{6,x} -$$

$$3kR_2 - Q_2 - \frac{1}{R}(S_{2,y} + kL_{2,y}) \Big) \delta \psi_y \Big\} \, dx dy dt$$

$$
\begin{aligned}
-\int_{t_1}^{t_2}\int_0^b\Big\{ & N_1\delta u_o + \left[N_6 + \frac{1}{2R}M_6\right]\delta v_o + \Big[-k(P_{1,x} + 2P_{6,y}) + Q_1 + \\
& 3kR_1 + \frac{1}{R}kL_{6,y} + \bar{N}_1 w_{,x} + \bar{N}_6 w_{,y}\Big]\delta w + (M_1 + 2kP_1)\delta\psi_x + \\
& (M_6 + kP_6)\delta\psi_y\Big\}\Bigg|_{x=0}^{x=a} dydt
\end{aligned}
$$

$$
\begin{aligned}
-\int_{t_1}^{t_2}\int_0^a\Big\{ & \left[N_6 - \frac{1}{2R}M_6\right]\delta u_o + N_2\delta v_o + \Big[-k(P_{2,y} + 2P_{6,x}) + Q_2 + \\
& 3kR_2 + \frac{1}{R}k(L_{2,y} + L_{6,x}) + \bar{N}_2 w_{,y} + \bar{N}_6 w_{,x}\Big]\delta w + \\
& \left[M_6 + kP_6 + \frac{1}{R}(-S_6 - kL_6)\right]\delta\psi_x + \\
& \left[M_2 + 2kP_2 + \frac{1}{R}(-S_2 - 2kL_2)\right]\delta\psi_y\Big\}\Bigg|_{y=0}^{y=b} dxdt
\end{aligned}
$$

$$
-\int_{t_1}^{t_2}\left\{k\left[2P_6 - \frac{1}{R}L_6\right]\delta w\right\}\Bigg|_{y=0}^{y=b}\Bigg|_{x=0}^{x=a} dt = 0 \qquad (2.32)
$$

The double integral over the domain in Eq (2.32) contains the five equations of motion. The two line integrals are the geometric and natural boundary conditions along the four edges of the shell panel, and the last term expresses the boundary conditions at the four corners. In the double integral, the variations of the degrees of freedom (δu_o, δv_o, δw, $\delta\psi_x$, and $\delta\psi_y$) are arbitrary and in general are not equal to zero. Consequently, their corresponding coefficients must equal zero, yielding the five coupled partial differential equations of motion for the panel at any time, t:

δu_o:

$$N_{1,x} + N_{6,y} - \frac{1}{2R}M_{6,y} = I_1\ddot{u}_o + \bar{I}_2\ddot{\psi}_x - \bar{I}_3\ddot{w}_{,x}$$

δv_o:

$$N_{2,y} + N_{6,x} + \frac{1}{2R}M_{6,x} = \bar{I}_1{}'\ddot{v}_o + \bar{I}_2{}'\ddot{\psi}_y - \bar{I}_3{}'\ddot{w}_{,y}$$

δw:

$$- k(P_{1,xx} + P_{2,yy} + 2P_{6,xy}) + Q_{2,y} + Q_{1,x} + 3k(R_{2,y} + R_{1,x}) -$$

$$\frac{1}{R}[N_2 - k(L_{2,yy} + L_{6,xy})] + \bar{N}_1 w_{,xx} + 2\bar{N}_6 w_{,xy} - \bar{N}_2\left[\frac{1}{R} - w_{,yy}\right]$$

$$= \bar{I}_3\ddot{u}_{o,x} + \bar{I}_5\ddot{\psi}_{x,x} + \bar{I}_3{}'\ddot{v}_{o,y} - k^2 I_7(\ddot{w}_{,xx} + \ddot{w}_{,yy}) + \bar{I}_5\ddot{\psi}_{y,y} + I_1\ddot{w}$$

$\delta\psi_x$:

$$k(P_{1,x} + P_{6,y}) + M_{1,x} + M_{6,y} - 3kR_1 - Q_1 - \frac{1}{R}(S_{6,y} + kL_{6,y})$$

$$= \bar{I}_2\ddot{u}_o + \bar{I}_4\ddot{\psi}_x - \bar{I}_5\ddot{w}_{,x}$$

$\delta\psi_y$:

$$k(P_{2,y} + P_{6,x}) + M_{2,y} + M_{6,x} - 3kR_2 - Q_2 - \frac{1}{R}(S_{2,y} + kL_{2,y})$$

$$= \bar{I}_2{}'\ddot{v}_o + \bar{I}_4\ddot{\psi}_y - \bar{I}_5\ddot{w}_{,y} \qquad (2.33)$$

These equations of motion will simplify to those of other authors for certain applications. If $R \rightarrow \infty$ in Eqs (2.21), (2.32), and (2.33), the equations of motion and boundary conditions reduce to those of a flat plate with parabolic transverse shear and rotary inertia. (See (14), (15), and (17).) If the following terms are *deleted* from the equations of motion in Eq (2.33):

$$\delta u_o: \quad -\frac{1}{2R} M_{6,y}$$

$$\delta v_o: \quad \frac{1}{2R} M_{6,x}$$

$$\delta w: \quad \frac{1}{R} k(L_{2,yy} + L_{6,xy})$$

$$\delta \psi_x: \quad -\frac{1}{R}(S_{6,y} + kL_{6,y})$$

$$\delta \psi_y: \quad -\frac{1}{R}(S_{2,y} + kL_{2,y}) \tag{2.33a}$$

the equations of motion reduce to the Donnell equations of motion as presented by Reddy (16). For $h/R \approx 1/50$, the terms in Eq (2.33a) are small relative to the other terms in Eq (2.33), thus establishing the 1/50 limit used by Reddy.

The general equations developed so far need to be tailored to meet the needs of the specific circular cylindrical shell panel to be considered in this thesis. First of all, this thesis will only consider symmetric laminates: that is, laminates that are symmetric about the mid surface with respect to both material properties and geometry (fiber orientation angle, θ_k, and thickness, t_k). Therefore, the following stiffness matrices from Eqs (2.16) and (2.17) will drop out (9):

$$\left[\ B_{ij}\right] = \left[\ E_{ij}\right] = \left[\ G_{ij}\right] = \left[\ I_{ij}\right] = \left[\ 0\ \right] \tag{2.33b}$$

Additionally, since ρ is constant with respect to z (all laminae have the same density), the inertia terms in Eq (2.21) may be integrated to yield:

$$I_2 = I_4 = \bar{I}_2 = \bar{I}_3 = 0$$

$$I_1 = \bar{I}_1{}' = \rho h$$

$$I_3 = \rho h^3/12, \; I_5 = \rho h^5/80, \; I_7 = \rho h^7/448 \tag{2.33c}$$

$$\bar{I}_2{}' = \frac{1}{R}\rho h^3/15, \; \bar{I}_3{}' = \frac{1}{R}\rho h^3/60, \; \bar{I}_4 = 17\rho h^3/315, \; \bar{I}_5 = 4\rho h^3/315$$

The last simplification concerns the acceleration terms in Eqs (2.32) and (2.33). This thesis will consider rotary inertia: not in-plane inertia. Consequently, the following in-plane acceleration terms will drop out: $\ddot{u}_o = \ddot{v}_o = \ddot{u}_{o,x} = \ddot{v}_{o,y} = 0$.

During the development of the kinetic energy, time dependant boundaries were ignored because this thesis is concerned exclusively with harmonic problems. Assuming harmonic solution forms and applying separation of variables, the five degrees of freedom and their corresponding accelerations may be expressed as:

$$u_o = u_o(x,y,t) = u_o(x,y)\sin\omega t$$
$$v_o = v_o(x,y,t) = v_o(x,y)\sin\omega t$$
$$w = w(x,y,t) = w(x,y)\sin\omega t \;, \; \ddot{w} = -\omega^2 w(x,y)\sin\omega t = -\omega^2 w$$
$$\psi_x = \psi_x(x,y,t) = \psi_x(x,y)\sin\omega t, \; \ddot{\psi}_x = -\omega^2\psi_x(x,y)\sin\omega t = -\omega^2\psi_x$$
$$\psi_y = \psi_y(x,y,t) = \psi_y(x,y)\sin\omega t, \; \ddot{\psi}_y = -\omega^2\psi_y(x,y)\sin\omega t = -\omega^2\psi_y$$

where ω is the natural frequency of vibration. (2.33d)

If these expressions are substituted into Hamilton's principle, Eq (2.32), bearing in mind that all the resultant quantities ($\{N_i\}$, $\{M_i\}$, $\{S_i\}$, $\{P_i\}$, $\{L_i\}$, $\{Q_i\}$, and $\{R_i\}$) are functions of

the *spatial* derivatives of u_o, v_o, w, ψ_x, and ψ_y, the term $\sin\omega t$ may be factored out from the entire expression and integrated with respect to time:

$$\int_{t_1}^{t_2} \sin\omega t dt = - \frac{1}{\omega}\cos\omega t \Big|_{t_1}^{t_2} \tag{2.33e}$$

The integrand may be canceled to the right side of the equation, leaving Eq (2.32) independent of time.

The concepts presented in these paragraphs are now incorporated into Hamilton's Principle. Equation (2.32) is partitioned into five equations. Each contains the equation of motion plus the associated boundary condition for a particular degree of freedom. (4), (21), and (23).

Equation (2.32) for u_o yields:

$$\int_0^b \int_0^a \left[N_{1,x} + N_{6,y} - \frac{1}{2R}M_{6,y}\right]\delta u_o dxdy +$$

$$\int_0^b N_1\delta u_o\Big|_{x=a}^{x=0} dy + \int_0^a \left[N_6 - \frac{1}{2R}M_6\right]\delta u_o\Big|_{y=b}^{y=0} dx = 0 \tag{2.34}$$

Equation (2.32) for v_o yields:

$$\int_0^b \int_0^a \left[\bar{I}_2{}'\omega^2\psi_y - \bar{I}_3{}'\omega^2 w_{,y} + N_{2,y} + N_{6,x} + \frac{1}{2R}M_{6,x}\right]\delta v_o dxdy +$$

$$\int_0^b \left[N_6 + \frac{1}{2R}M_6\right]\delta v_o\Big|_{x=a}^{x=0} dy + \int_0^a N_2\delta v_o\Big|_{y=b}^{y=0} dx = 0 \tag{2.35}$$

Equation (2.32) for w yields:

$$\int_0^b \int_0^a \Big[\bar{I}_5 \omega^2 (\psi_{x,x} + \psi_{y,y}) - k^2 I_7 \omega^2 (w,_{xx} + w,_{yy}) + I_1 \omega^2 w + Q_{1,x} -$$

$$k(P_{1,xx} + P_{2,yy} + 2P_{6,xy}) + Q_{2,y} + 3k(R_{2,y} + R_{1,x}) -$$

$$\frac{1}{R}[N_2 - k(L_{2,yy} + L_{6,xy})] + \bar{N}_1 w,_{xx} + 2\bar{N}_6 w,_{xy} -$$

$$\bar{N}_2 \left(\frac{1}{R} - w,_{yy} \right) \Big] \delta w \, dxdy +$$

$$\int_0^b \Big\{ -k(P_{1,x} + 2P_{6,y}) + Q_1 + 3kR_1 + \frac{1}{R} kL_{6,y}$$

$$+ \bar{N}_1 w,_x + \bar{N}_6 w,_y \Big\} \delta w \Big|_{x=a}^{x=0} dy +$$

$$\int_0^a \Big\{ -k(P_{2,y} + 2P_{6,x}) + Q_2 + 3kR_2 + \frac{1}{R} k(L_{2,y} + L_{6,x})$$

$$+ \bar{N}_2 w,_y + \bar{N}_6 w,_x \Big\} \delta w \Big|_{y=b}^{y=0} dx +$$

$$k\left\{ 2P_6 - \frac{1}{R} L_6 \right\} \delta w \Big\} \Big|_{y=b}^{y=0} \Big|_{x=a}^{x=0} = 0 \qquad (2.36)$$

Equation (2.32) for ψ_x yields:

$$\int_0^b \int_0^a \Big[\bar{I}_4\omega^2\psi_x - \bar{I}_5\omega^2 w_{,x} + k(P_{1,x} + P_{6,y}) + M_{1,x} + M_{6,y} - Q_1 -$$

$$3kR_1 - \frac{1}{R}(S_{6,y} + kL_{6,y})\Big]\delta\psi_x dxdy +$$

$$\int_0^b \Big[M_1 + 2kP_1\Big]\delta\psi_x \Big|_{x=a}^{x=0} dy +$$

$$\int_0^a \Big[M_6 + kP_6 + \frac{1}{R}(-S_6 - kL_6)\Big]\delta\psi_x \Big|_{y=b}^{y=0} dx = 0 \tag{2.37}$$

And, finally Eq (2.32) for ψ_y yields:

$$\int_0^b \int_0^a \Big[\bar{I}_4\omega^2\psi_y - \bar{I}_5\omega^2 w_{,y} + k(P_{2,y} + P_{6,x}) + M_{2,y} + M_{6,x} - Q_2 -$$

$$3kR_2 - \frac{1}{R}(S_{2,y} + kL_{2,y})\Big]\delta\psi_y dxdy +$$

$$\int_0^b \Big[M_6 + kP_6\Big]\delta\psi_x \Big|_{x=a}^{x=0} dy +$$

$$\int_0^a \Big[M_2 + 2kP_2 + \frac{1}{R}(-S_2 - 2kL_2)\Big]\delta\psi_y \Big|_{y=b}^{y=0} dx = 0 \tag{2.38}$$

The final step in the development of the equations of motion and boundary conditions is to substitute the resultant quantities in Eq (2.17) and the strain-displacement relations in Eqs (2.6) and (2.7) into Eqs (2.34) through (2.38). With the aid of MACSYMA (25) to perform the extensive algebraic manipulations, these five equations may be expressed in terms of the degrees of freedom and stiffness terms.

Equation (2.34) for u_o becomes:

$$\int_0^b \int_0^a \Big\{ A_{66}u_{o,yy} + A_{11}u_{o,xx} + 2A_{16}u_{o,xy} + A_{26}v_{o,yy} + A_{16}v_{o,xx} +$$

$$(A_{66} + A_{12})v_{o,xy} + \frac{1}{R}\Big[- \frac{3}{2}\, kF_{26}w,_{yyy} + A_{26}w,_{y} + A_{12}w,_{x} -$$

$$\frac{3}{2}\, kF_{16}w,_{xxy} -k(2F_{66} + F_{12})w,_{xyy} - \frac{3}{2}\,(kF_{66} + D_{66})\psi_{x,yy} -$$

$$\frac{3}{2}\,(kF_{16} + D_{16})\psi_{x,xy} - \frac{3}{2}\,(kF_{26} + D_{26})\psi_{y,yy} - (\frac{1}{2}\, kF_{66} +$$

$$kF_{12} + \frac{1}{2}\, D_{66} + D_{12})\psi_{y,xy} - \frac{1}{R}\,\frac{1}{4}\, D_{66}(v_{o,xy} - u_{o,yy})\Big]\Big\}\delta u_o dxdy$$

$$+ \int_0^b \Big\{ A_{11}u_{o,x} + A_{16}u_{o,y} + A_{12}v_{o,y} + A_{16}v_{o,x} + \frac{1}{R}\Big[-kF_{12}w,_{yy} -$$

$$kF_{16}w,_{xy} + A_{12}w -(kF_{12} + D_{12})\psi_{y,y} - (kF_{16} + D_{16})\psi_{x,y}\Big]\Big\}\delta u_o\Big|_{x=a}^{x=0} dy$$

$$+ \int_0^a \Big\{ A_{16}u_{o,x} + A_{66}u_{o,y} + A_{26}v_{o,y} + A_{66}v_{o,x} + \frac{1}{R}\Big[- \frac{3}{2}\, kF_{26}w,_{yy} -$$

$$\frac{1}{2}\, kF_{16}w,_{xx} - 2kF_{66}w,_{xy} + A_{26}w - \frac{3}{2}\,(kF_{26} + D_{26})\psi_{y,y} -$$

$$\frac{1}{2}\,(kF_{66} + D_{66})\psi_{y,x} - \frac{3}{2}\,(kF_{66} + D_{66})\psi_{x,y} - \frac{1}{2}\,(kF_{16} + D_{16})\psi_{x,x}$$

$$- \frac{1}{R}\,\frac{1}{4}\, D_{66}(v_{o,x} - u_{o,y})\Big]\Big\}\delta u_o\Big|_{y=b}^{y=0} dx = 0 \qquad (2.39)$$

Equation (2.35) for v_o becomes:

$$\int_0^b \int_0^a \Big\{ \bar{I}_2{}' \omega^2 \psi_y - \bar{I}_3{}' \omega^2 w,_y + A_{16} u_{o,xx} + (A_{66} + A_{12}) u_{o,xy} + A_{26} u_{o,yy}$$

$$+ A_{22} v_{o,yy} + A_{66} v_{o,xx} + 2A_{26} v_{o,xy} + \frac{1}{R}\Big[-kF_{22} w,_{yyy} + A_{22} w,_y$$

$$+ \frac{1}{2} kF_{16} w,_{xxx} - \frac{3}{2} kF_{26} w,_{xyy} + A_{26} w,_x - (kF_{26} + D_{26}) \psi_{x,yy}$$

$$+ \frac{1}{2} (kF_{16} + D_{16}) \psi_{x,xx} - \frac{1}{2} (kF_{66} + D_{66}) \psi_{x,xy} - (kF_{22} +$$

$$D_{22}) \psi_{y,yy} + \frac{1}{2} (kF_{66} + D_{66}) \psi_{y,xx} - \frac{1}{2} (kF_{26} + D_{26}) \psi_{y,xy} -$$

$$\frac{1}{R} \frac{1}{4} D_{66} (u_{o,xy} - v_{o,xx}) \Big] \Big\} \delta v_o \, dx dy$$

$$+ \int_0^b \Big\{ A_{16} u_{o,x} + A_{66} u_{o,y} + A_{66} v_{o,x} + A_{26} v_{o,y} + \frac{1}{R}\Big[- \frac{1}{2} kF_{26} w,_{yy} +$$

$$\frac{1}{2} kF_{16} w,_{xx} + A_{26} w - \frac{1}{2} (kF_{66} + D_{66}) \psi_{x,y} + \frac{1}{2} (kF_{16} +$$

$$+ D_{16}) \psi_{x,x} - \frac{1}{2} (kF_{26} + D_{26}) \psi_{y,y} + \frac{1}{2} (kF_{66} + D_{66}) \psi_{y,x} -$$

$$\frac{1}{R} \frac{1}{4} D_{66} (u_{o,y} - v_{o,x}) \Big] \Big\} \delta v_o \Big|_{x=a}^{x=0} dy$$

$$+ \int_0^a \Big\{ A_{12} u_{o,x} + A_{26} u_{o,y} + A_{26} v_{o,x} + A_{22} v_{o,y} + \frac{1}{R}\Big[-kF_{22} w,_{yy} -$$

$$kF_{26} w,_{xy} + A_{22} w - (kF_{26} + D_{26}) \psi_{x,y} -$$

$$(kF_{22} + D_{22}) \psi_{y,y} \Big] \Big\} \delta v_o \Big|_{y=b}^{y=0} dx = 0 \qquad (2.40)$$

Equation (2.36) for w becomes:

$$\int_0^b \int_0^a \Big\{ \bar{I}_5\omega^2(\psi_{x,x} + \psi_{y,y}) - k^2 I_7\omega^2(w,_{xx} + w,_{yy}) + I_1\omega^2 w + \bar{N}_1 w,_{xx} +$$

$$2\bar{N}_6 w,_{xy} + \bar{N}_2 w,_{yy} - k^2 H_{22} w,_{yyyy} + (6kD_{44} + 9k^2F_{44} + A_{44}) w,_{yy}$$

$$- k^2 H_{11} w,_{xxxx} - 4k^2 H_{16} w,_{xxxy} - 2k^2(2H_{66} + H_{12}) w,_{xxyy} +$$

$$(6kD_{55} + 9k^2 + A_{55}) w,_{xx} - 4k^2 H_{26} w,_{xyyy} + (12kD_{45} + 18k^2F_{45}$$

$$+ 2A_{45}) w,_{xy} - (k^2H_{26} + kF_{26})\psi_{x,yyy} + (6kD_{45} + 9k^2F_{45} +$$

$$A_{45})\psi_{x,y} - (k^2H_{11} + kF_{11})\psi_{x,xxx} - (3k^2H_{16} + 3kF_{16})\psi_{x,xxy} -$$

$$k(2kH_{66} + kH_{12} + 2F_{66} + F_{12})\psi_{x,xyy} + (6kD_{55} + 9k^2F_{55} +$$

$$A_{55})\psi_{x,x} - k(kH_{22} + F_{22})\psi_{y,yyy} + (6kD_{44} + 9k^2F_{44} + A_{44})\psi_{y,y}$$

$$-k(kH_{16} + F_{16})\psi_{y,xxx} - k(2kH_{66} + kH_{12} + 2F_{66} + F_{12})\psi_{y,xxy}$$

$$-k(3kH_{26} + 3F_{26})\psi_{y,xyy} + (6kD_{45} + 9k^2F_{45} + A_{45})\psi_{y,x} +$$

$$\frac{1}{R}\Big[-\bar{N}_2 + \frac{3}{2}kF_{26}u_{o,yyy} - A_{26}u_{o,y} + \frac{3}{2}kF_{16}u_{o,xxy} - A_{12}u_{o,x}$$

$$+ k(2F_{66} + F_{12})u_{o,xyy} + kF_{22}v_{o,yyy} - A_{22}v_{o,y} - A_{26}v_{o,x} -$$

$$\frac{1}{2}kF_{16}v_{o,xxx} + \frac{3}{2}kF_{26}v_{o,xyy} + \frac{1}{R}\Big(-k^2J_{22}w,_{yyyy} + 2kF_{22}w,_{yy}$$

$$-k^2J_{66}w,_{xxyy} - 2k^2J_{26}w,_{xyyy} + 2kF_{26}w,_{xy} - A_{22}w - k(kJ_{26} +$$

$$H_{26})\psi_{x,yyy} + (kF_{26} + D_{26})\psi_{x,y} - k(kJ_{66} + H_{66})\psi_{x,xyy} -$$

$$k(kJ_{22} + H_{22})\psi_{y,yyy} + (kF_{22} + D_{22})\psi_{y,y} -$$

$$k(kJ_{26} + H_{26})\psi_{y,xyy}\Big)\Big]\Big\}\delta w dxdy$$

$$+ \int_0^b \Big\{ \bar{N}_1 w,_x + \bar{N}_6 w,_y - 2k^2 H_{26} w,_{yyy} + (6kD_{45} + 9k^2 F_{45} + A_{45}) w,_y -$$

$$k^2 H_{11} w,_{xxx} - 4k^2 H_{16} w,_{xxy} + (6kD_{55} + 9k^2 F_{55} + A_{55}) w,_x -$$

$$k^2 (4H_{66} + H_{12}) w,_{xyy} - 2k(kH_{26} + F_{26}) \psi_{y,yy} - k(kH_{16} + F_{16}) \psi_{y,xx}$$

$$- k(2kH_{66} + kH_{12} + 2F_{66} + F_{12}) \psi_{y,xy} + (6kD_{45} + 9k^2 F_{45} +$$

$$A_{45}) \psi_y - 2k(kH_{66} + F_{66}) \psi_{x,yy} - k(kH_{11} + F_{11}) \psi_{x,xx} - 3k(kH_{16}$$

$$+ F_{16}) \psi_{x,xy} + (6kD_{55} + 9k^2 F_{55} + A_{55}) \psi_x + \frac{1}{R} \Big[2kF_{66} u_{o,yy} +$$

$$\frac{3}{2} kF_{16} u_{o,xy} + kF_{26} v_{o,yy} - \frac{1}{2} kF_{16} v_{o,xx} + \frac{1}{R} \Big\{ -k^2 J_{26} w,_{yyy}$$

$$+ kF_{26} w,_y - k^2 J_{66} w,_{xyy} - k(kJ_{66} + H_{66}) \psi_{x,yy}$$

$$- k(kJ_{26} + H_{26}) \psi_{y,yy} \Big\} \Big] \Big\} \delta w \, \Big|_{x=a}^{x=0} dy$$

$$+ \int_0^a \Big\{ \bar{N}_2 w,_y + \bar{N}_6 w,_x - k^2 H_{22} w,_{yyy} + (6kD_{44} + 9k^2 F_{44} + A_{44}) w,_y -$$

$$2k^2 H_{16} w,_{xxx} - k^2 (4H_{66} + H_{12}) w,_{xxy} - 4k^2 H_{26} w,_{xyy} + (6kD_{45} +$$

$$9k^2 F_{45} + A_{45}) w,_x - k(kH_{26} + F_{26}) \psi_{x,yy} - 2k(kH_{16} + F_{16}) \psi_{x,xx}$$

$$-k(2kH_{66} + kH_{12} + 2F_{66} + F_{12}) \psi_{x,xy} + (6kD_{45} + 9k^2 F_{45} + A_{45}) \psi_x$$

$$-k(kH_{22} + F_{22}) \psi_{y,yy} - 2k(kH_{66} + F_{66}) \psi_{y,xx} - 3k(kH_{26} + F_{26}) \psi_{y,xy}$$

$$+(6kD_{44} + 9k^2 F_{44} + A_{44}) \psi_y + \frac{1}{R} \Big[kF_{16} u_{o,xx} + \frac{3}{2} kF_{26} u_{o,yy} +$$

$$k(2F_{66} + F_{12}) u_{o,xy} + kF_{22} v_{o,yy} + \frac{3}{2} kF_{26} v_{o,xy} + \frac{1}{R} \Big\{ kF_{22} w,_y$$

$$-k^2 J_{22} w,_{yyy} - k^2 J_{66} w,_{xxy} - 2k^2 J_{26} w,_{xyy} + kF_{26} w,_x - k(kJ_{26} +$$

$$H_{26}) \psi_{x,yy} - k(kJ_{66} + H_{66}) \psi_{x,xy} - k(kJ_{22} + H_{22}) \psi_{y,yy} - k(kJ_{26} +$$

$$H_{26})\psi_{y,xy}\Big]\Big]\Big\}\delta w\,\Big|_{y=b}^{y=0}dx$$

$$+\ \Big\{2k^2H_{26}w_{,yy} + 2k^2H_{16}w_{,xx} + 4k^2H_{66}w_{,xy} + 2k(kH_{66} + F_{66})\psi_{x,y}$$

$$+ 2k(kH_{16} + F_{16})\psi_{x,x} + 2k(kH_{26} + F_{26})\psi_{y,y} + 2k(kH_{66} +$$

$$F_{66})\psi_{y,x} + \frac{1}{R}\Big[-2kF_{66}u_{o,y} - kF_{16}u_{o,x} - kF_{26}v_{o,y} + \frac{1}{R}\Big(-kF_{26}w$$

$$+ k^2J_{26}w_{,yy} + k^2J_{66}w_{,xy} + k(kJ_{26} + H_{26})\psi_{y,y}$$

$$+ k(kJ_{66} + H_{66})\psi_{x,y}\Big]\Big]\Big\}\delta w\Big|_{y=b}^{y=0}\Big|_{x=a}^{x=0} = 0 \qquad (2.41)$$

Equation (2.37) for ψ_x becomes:

$$\int_0^b \int_0^a \Big\{ \bar{I}_4\omega^2\psi_x - \bar{I}_5\omega^2 w,_x + k(kH_{26} + F_{26})w,_{yyy} + k(kH_{11} + F_{11})w,_{xxx}$$
$$- (6kD_{45} + 9k^2F_{45} + A_{45})w,_y + 3k(kH_{16} + F_{16})w,_{xxy} + k(F_{12}$$
$$+ 2kH_{66} + kH_{12} + 2F_{66})w,_{xyy} - (6kD_{55} + 9k^2F_{55} + A_{55})w,_x +$$
$$(k^2H_{66} + 2kF_{66} + D_{66})\psi_{x,yy} + (k^2H_{11} + 2kF_{11} + D_{11})\psi_{x,xx} +$$
$$2(k^2H_{16} + 2kF_{16} + D_{16})\psi_{x,xy} - (6kD_{55} + 9k^2F_{55} + A_{55})\psi_x +$$
$$(k^2H_{26} + 2kF_{26} + D_{26})\psi_{y,yy} + (k^2H_{16} + 2kF_{16} + D_{16})\psi_{y,xx} +$$
$$(k^2H_{66} + k^2H_{12} + 2kF_{66} + 2kF_{12} + D_{66} + D_{12})\psi_{y,xy} - (6kD_{45}$$
$$+ 9k^2F_{45} + A_{45})\psi_y + \frac{1}{R}\Big[- \frac{3}{2}(kF_{66} + D_{66})u_{o,yy} - \frac{3}{2}(kF_{16} +$$
$$D_{16})u_{o,xy} - (kF_{26} + D_{26})v_{o,yy} + \frac{1}{2}(kF_{16} + D_{16})v_{o,xx} -$$
$$\frac{1}{2}(kF_{66} + D_{66})v_{o,xy} + \frac{1}{R}\Big\{k(kJ_{26} + H_{26})w,_{yyy} - (kF_{26} +$$
$$D_{26})w,_y + k(kJ_{66} + H_{66})w,_{xyy} + (k^2J_{66} + 2kH_{66} + F_{66})\psi_{x,yy}$$
$$+ (k^2J_{26} + 2kH_{26} + F_{26})\psi_{y,yy}\Big\}\Big]\Big\}\delta\psi_x dxdy$$

$$+ \int_0^b \Big\{ k(2kH_{12} + F_{12})w,_{yy} + k(2kH_{11} + F_{11})w,_{xx} + 2k(2kH_{16} + F_{16})w,_{xy}$$
$$+ (2k^2H_{16} + 3kF_{16} + D_{16})\psi_{x,y} + (2k^2H_{11} + 3kF_{11} + D_{11})\psi_{x,x}$$
$$+ (2k^2H_{12} + 3kF_{12} + D_{12})\psi_{y,y} + (2k^2H_{16} + 3kF_{16} + D_{16})\psi_{y,x}$$
$$+ \frac{1}{R}\Big\{-(kF_{16} + \frac{1}{2}D_{16})u_{o,y} + (kF_{16} + \frac{1}{2}D_{16})v_{o,x}\Big\}\Big\}\delta\psi_x \Big|_{x=a}^{x=0} dy$$

$$
\begin{aligned}
&+ \int_0^a \Big\{ k(kH_{26} + F_{26})w,_{yy} + k(kH_{16} + F_{16})w,_{xx} + 2k(kH_{66} + F_{66})w,_{xy} \\
&\quad + (k^2H_{66} + 2kF_{66} + D_{66})\psi_{x,y} + (k^2H_{16} + 2kF_{16} + D_{16})\psi_{x,x} + \\
&\quad (k^2H_{26} + 2kF_{26} + D_{26})\psi_{y,y} + (k^2H_{66} + 2kF_{66} + D_{66})\psi_{y,x} + \\
&\quad \frac{1}{R}\Big[- \frac{3}{2} (kF_{66} + D_{66})u_{o,y} - (kF_{16} + D_{16})u_{o,x} - (kF_{26} + \\
&\quad D_{26})v_{o,y} - \frac{1}{2} (kF_{66} + D_{66})v_{o,x} + \frac{1}{R}\Big(k(kJ_{26} + H_{26})w,_{yy} + \\
&\quad k(kJ_{66} + H_{66})w,_{xy} - (kF_{26} + D_{26})w + (k^2J_{66} + 2kH_{66} + F_{66})\psi_{x,y} \\
&\quad + (k^2J_{26} + 2kH_{26} + F_{26})\psi_{y,y}\Big)\Big]\Big\}\delta\psi_x \Big|_{y=b}^{y=0} dx \quad = \quad 0 \qquad (2.42)
\end{aligned}
$$

And, finally, Eq (2.38) for ψ_y becomes:

$$\int_0^b \int_0^a \Big\{ \bar{I}_4\omega^2\psi_y - \bar{I}_5\omega^2 w,_y + k(kH_{22} + F_{22})w,_{yyy} - (6kD_{44} + 9k^2F_{44} + A_{44})w,_y + k(kH_{16} + F_{16})w,_{xxx} + k(2kH_{66} + kH_{12} + 2F_{66} + F_{12})w,_{xxy} + 3k(kH_{26} + F_{26})w,_{xyy} - (6kD_{45} + 9k^2F_{45} + A_{45})w,_x + (k^2H_{26} + 2kF_{26} + D_{26})\psi_{x,yy} + (k^2H_{16} + 2kF_{16} + D_{16})\psi_{x,xx} + (k^2H_{66} + k^2H_{12} + 2kF_{66} + 2kF_{12} + D_{66} + D_{12})\psi_{x,xy} - (6kD_{45} + 9k^2F_{45} + A_{45})\psi_x + (k^2H_{22} + 2kF_{22} + D_{22})\psi_{y,yy} + (k^2H_{66} + 2kF_{66} + D_{66})\psi_{y,xx} + 2(k^2H_{26} + 2kF_{26} + D_{26})\psi_{y,xy} - (6kD_{44} + 9k^2F_{44} + A_{44})\psi_y + \frac{1}{R}\Big[- \frac{3}{2}(kF_{26} + D_{26})u_{o,yy} - \frac{1}{2}(kF_{66} + 2kF_{12} + D_{66} + 2D_{12})u_{o,xy} - (kF_{22} + D_{22})v_{o,yy} + \frac{1}{2}(kF_{66} + D_{66})v_{o,xx} - \frac{1}{2}(kF_{26} + D_{26})v_{o,xy} + \frac{1}{R}\Big\{k(kJ_{22} + H_{22})w,_{yyy} - (kF_{22} + D_{22})w,_y + k(kJ_{26} + H_{26})w,_{xyy} + (k^2J_{22} + 2kH_{22} + F_{22})\psi_{y,yy} + (k^2J_{26} + 2kH_{26} + F_{26})\psi_{x,yy}\Big\}\Big]\Big\}\delta\psi_y dxdy$$

$$+ \int_0^b \Big\{ k(kH_{26} + F_{26})w,_{yy} + k(kH_{16} + F_{16})w,_{xx} + 2k(kH_{66} + F_{66})w,_{xy} + (k^2H_{26} + 2kF_{26} + D_{26})\psi_{y,y} + (k^2H_{66} + 2kF_{66} + D_{66})\psi_{y,x} + (k^2H_{66} + 2kF_{66} + D_{66})\psi_{x,y} + (k^2H_{16} + 2kF_{16} + D_{16})\psi_{x,x} + \frac{1}{R}\Big[\frac{1}{2}(kF_{66} + D_{66})v_{o,x} - \frac{1}{2}(kF_{66} + D_{66})u_{o,y}\Big]\Big\}\delta\psi_y \Big|_{x=a}^{x=0} dy$$

$$
\begin{aligned}
+ \int_0^a \Big\{ & k(2kH_{22} + F_{22})w,_{yy} + k(2kH_{12} + F_{12})w,_{xx} + 2k(2kH_{26} + F_{26})w,_{xy} \\
& + (2k^2H_{26} + 3kF_{26} + D_{26})\psi_{x,y} + (2k^2H_{12} + 3kF_{12} + D_{12})\psi_{x,x} \\
& + (2k^2H_{22} + 3kF_{22} + D_{22})\psi_{y,y} + (2k^2H_{26} + 3kF_{26} + D_{26})\psi_{y,x} \\
& + \frac{1}{R}\Big[- \frac{3}{2}(2kF_{26} + D_{26})u_{o,y} - (2kF_{12} + D_{12})u_{o,x} - (2kF_{22} + \\
& D_{22})v_{o,y} - \frac{1}{2}(2kF_{26} + D_{26})v_{o,x} + \frac{1}{R}\Big\{k(2kJ_{22} + H_{22})w,_{yy} + \\
& k(2kJ_{26} + H_{26})w,_{xy} - (2kF_{22} + D_{22})w + (2k^2J_{22} + 3kH_{22} + \\
& F_{22})\psi_{y,y} + (2k^2J_{26} + 3kH_{26} + F_{26})\psi_{x,y}\Big\}\Big]\Big\}\delta\psi_y\Big|_{y=b}^{y=0} dx = 0
\end{aligned}
\tag{2.43}
$$

Several observations can be made concerning Eqs (2.39) to (2.43). For a flat plate, small deflection theory dictates that bending displacement is completely decoupled from membrane displacement. If the radius of curvature, R, approaches infinity, the five circular cylindrical shell panel equations reduce to those of a flat plate. The two membrane equations for u_o , Eq (2.39), and v_o, Eq (2.40), will consist only of extensional stiffness terms, A_{ij}, and spatial derivatives of u_o and v_o, as expected. Additionally, the three bending equations for w, ψ_x, and ψ_y in Eqs (2.41), (2.42), and (2.43) will consist only of bending and higher order stiffness terms and spatial derivatives of w, ψ_x, and ψ_y. If the higher order stiffness terms are dropped, leaving only A_{ij} and D_{ij}, the three flat plate equations will reduce to those of (3), (4), and (20), which were obtained from the lower order Mindlin transverse shear strain modeling. For R not equal to infinity, membrane

and bending displacement are coupled. To find the natural frequencies and buckling loads of the circular cylindrical shell panel, all five equations must be solved simultaneously. This solution will be approximated using the Galerkin technique discussed in the next section.

GALERKIN TECHNIQUE

The Galerkin technique is an approximation technique that will be used to solve the five coupled partial differential equations. The general concepts of the method will be presented first, and then its specific application to this problem will be addressed. Most of the concepts presented here are from (11), (20), and (22).

The classic Galerkin technique works directly with the equation of motion of a particular system. For example, consider the following system (the same form as Eqs (2.39) to (2.43)):

$$\int_y \int_x DEOM(\xi(x,y))\delta\xi(x,y)dxdy + \int_y BC1(\xi(x,y))\delta\xi(x,y)\Big|_x dy + \int_x BC2(\xi(x,y))\delta\xi(x,y)\Big|_y dx = 0 \qquad (2.44)$$

where $\xi(x,y)$ is the degree of freedom, $DEOM(\xi(x,y))$ is the differential equation of motion that is a function of $\xi(x,y)$ and its spatial derivatives, and $BC1(\xi(x,y))$ and $BC2(\xi(x,y))$ are the associated boundary conditions, also a function of $\xi(x,y)$. The approximate solution has the form:

$$\xi(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn}\phi_{mn}(x,y) \qquad (2.45)$$

where A_{mn} are unknown constants to be determined later and $\phi_{mn}(x,y)$ are known linearly independent comparison functions, ie, functions that satisfy both the geometric and natural

(force) boundary conditions of the system. (11) If, say, the required degree of accuracy calls for M = N = 2, then Eq (2.45) yields:

$$\zeta(x,y) = A_{11}\phi_{11}(x,y) + A_{12}\phi_{12}(x,y) + A_{21}\phi_{21}(x,y) + A_{22}\phi_{22}(x,y) \tag{2.46a}$$

and its corresponding variation from (22):

$$\begin{aligned}\delta\zeta(x,y) &= \frac{\partial\zeta(x,y)}{\partial A_{11}}\delta A_{11} + \frac{\partial\zeta(x,y)}{\partial A_{12}}\delta A_{12} + \frac{\partial\zeta(x,y)}{\partial A_{21}}\delta A_{21} + \frac{\partial\zeta(x,y)}{\partial A_{22}}\delta A_{22} \\ &= \phi_{11}(x,y)\delta A_{11} + \phi_{12}(x,y)\delta A_{12} + \phi_{21}(x,y)\delta A_{21} + \phi_{22}(x,y)\delta A_{22}\end{aligned} \tag{2.46b}$$

Since $\phi_{mn}(x,y)$ are comparison functions, the boundary conditions in Eq (2.44) need not be considered. Therefore, Eqs (2.46a) and (2.46b) are substituted directly into the equation of motion only:

$$\int_y\int_x DEOM(\zeta(x,y))\delta\zeta(x,y)dxdy = 0$$

The following results:

$$\int_y\int_x\Big[DEOM(A_{11}\phi_{11}(x,y) + A_{12}\phi_{12}(x,y) + A_{21}\phi_{21}(x,y) + A_{22}\phi_{22}(x,y))\Big]\cdot$$

$$\Big[\phi_{11}(x,y)\delta A_{11} + \phi_{12}(x,y)\delta A_{12} + \phi_{21}(x,y)\delta A_{21} + \phi_{22}(x,y)\delta A_{22}\Big]dxdy = 0 \tag{2.47a}$$

or

$$\int_y\int_x \Big[\mathrm{DEOM}(A_{11}\phi_{11}(x,y) + A_{12}\phi_{12}(x,y) + A_{21}\phi_{21}(x,y) + A_{22}\phi_{22}(x,y))\Big]\cdot$$
$$\phi_{11}(x,y)\delta A_{11}dxdy +$$

$$\int_y\int_x \Big[\mathrm{DEOM}(A_{11}\phi_{11}(x,y) + A_{12}\phi_{12}(x,y) + A_{21}\phi_{21}(x,y) + A_{22}\phi_{22}(x,y))\Big]\cdot$$
$$\phi_{12}(x,y)\delta A_{12}dxdy +$$

$$\int_y\int_x \Big[\mathrm{DEOM}(A_{11}\phi_{11}(x,y) + A_{12}\phi_{12}(x,y) + A_{21}\phi_{21}(x,y) + A_{22}\phi_{22}(x,y))\Big]\cdot$$
$$\phi_{21}(x,y)\delta A_{21}dxdy +$$

$$\int_y\int_x \Big[\mathrm{DEOM}(A_{11}\phi_{11}(x,y) + A_{12}\phi_{12}(x,y) + A_{21}\phi_{21}(x,y) + A_{22}\phi_{22}(x,y))\Big]\cdot$$
$$\phi_{22}(x,y)\delta A_{22}dxdy = 0 \qquad (2.47b)$$

Since the variations of the constants, δA_{11}, δA_{12}, δA_{21}, and δA_{22} are arbitrary, the only way Eq (2.47) can be identically zero is that each integral go to zero individually. Thus, after canceling the variation to the right side, Eq (2.47) becomes:

$$\int_x\int_y \Big[\mathrm{DEOM}(A_{11}\phi_{11}(x,y) + A_{12}\phi_{12}(x,y) + A_{21}\phi_{21}(x,y) + A_{22}\phi_{22}(x,y))\Big]\cdot$$
$$\phi_{11}(x,y)dxdy = 0$$

$$\int_x \int_y \left[DEOM(A_{11}\phi_{11}(x,y) + A_{12}\phi_{12}(x,y) + A_{21}\phi_{21}(x,y) + A_{22}\phi_{22}(x,y)) \right] \cdot \phi_{12}(x,y)dxdy = 0$$

$$\int_y \int_x \left[DEOM(A_{11}\phi_{11}(x,y) + A_{12}\phi_{12}(x,y) + A_{21}\phi_{21}(x,y) + A_{22}\phi_{22}(x,y)) \right] \cdot \phi_{21}(x,y)dxdy = 0$$

$$\int_x \int_y \left[DEOM(A_{11}\phi_{11}(x,y) + A_{12}\phi_{12}(x,y) + A_{21}\phi_{21}(x,y) + A_{22}\phi_{22}(x,y)) \right] \cdot \phi_{22}(x,y)dxdy = 0 \qquad (2.48)$$

The single equation of motion in Eq (2.44) has been transformed into four equations which must be solved simultaneously to obtain A_{11}, A_{12}, A_{21}, and A_{22}. A key characteristic to note here is the variation of the degree of freedom, $\delta\xi(x,y)$, in each of the four equations has been replaced with a single term of the approximate series for $\xi(x,y)$ in Eq (2.46). In general there will be (M x N) terms in each equation and (M x N) equations, depending upon the degree of accuracy chosen in Eq (2.45).

Galerkin's technique will now be applied to the specific problem of this thesis. There are two major differences between this problem and the classic problem presented in Eqs (2.44) through (2.48). First, there are five coupled partial differential equations rather than just one. To account for this all five degrees of freedom assume approximate solution forms as shown (4):

$$\psi_x(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn}\bar{\psi}_{xmn}(x,y)$$

$$\psi_y(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} B_{mn}\bar{\psi}_{ymn}(x,y)$$

$$w(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn}\bar{w}_{mn}(x,y)$$

$$u_o(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} E_{mn}\bar{u}_{omn}(x,y)$$

$$v_o(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} G_{mn}\bar{v}_{omn}(x,y) \tag{2.49}$$

where, as before, A_{mn}, B_{mn}, C_{mn}, E_{mn}, and G_{mn} are unknown constants to be determined. The second difference is a fundamental departure from the classic Galerkin technique, and follows the same line of reasoning presented in (3), (21), and (23).

By examining Eqs (2.39) to (2.43), it is obvious there are very complicated natural boundary conditions. It would be virtually impossible to choose comparison functions to approximate the series in Eq (2.49). Alternatively, $\bar{\psi}_{xmn}(x,y)$, $\bar{\psi}_{ymn}(x,y)$, $\bar{w}_{mn}(x,y)$, $\bar{u}_{omn}(x,y)$, and $\bar{v}_{omn}(x,y)$ are chosen to be admissible functions: functions that satisfy only the geometric boundary conditions. As a consequence the Galerkin technique will be applied to the boundary conditions as well as to the equations of motion. For the example in Eq (2.44), this means the line integrals for the boundary conditions are included along with the double integral for the equation of motion when the Galerkin equations in Eq (2.48) are generated. The boundary conditions are treated the same way as the equation of motion:

replace $\xi(x,y)$ in $BC1(\xi(x,y))$ and $BC2(\xi(x,y))$ with the approximate solution in Eq (2.46), and replace $\delta\xi$ in each line integral with a single term in the approximation series. (Note that if it were possible to choose comparison functions for the five equations in Eqs (2.39) to (2.43), which fundamentally satisfied the natural boundary conditions, then only the equations of motion, ie, the double integrals, would have to be dealt with.)

With the general concepts in hand, the Galerkin technique will now be applied to the particular boundary conditions used in this thesis.

SIMPLY-SUPPORTED BOUNDARY CONDITION

In this section the admissible functions are found that satisfy simply supported boundary conditions on all four edges of the circular cylindrical shell panel. These functions will then be inserted into Eqs (2.39) to (2.43), and then the equations will be integrated. The equations will then be ready for the eigenvalue formulation and the subsequent determination of the natural frequencies and buckling loads.

For the panel simply supported on all sides, the following bending boundary conditions exist:

At $x = 0$ and $x = a$

$$w = \psi_y = 0$$

and

At $y = 0$ and $y = b$

$$w = \psi_x = 0$$

As Jones (9) states, there are four kinds of membrane simply supported boundary conditions possible. An S-2 type condition is used here such that at an edge of the panel, the normal displacement is not zero and the tangential displacement is zero:

At $x = 0$ and $x = a$

$$u_o \neq 0 \text{ and } v_o = 0$$

and

At $y = 0$ and $y = b$

$$u_o = 0 \text{ and } v_o \neq 0$$

Therefore, the admissible functions in Eq (2.49) become:

$$\psi_x(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn}\cos(m\pi x/a)\ \sin(n\pi y/b)$$

$$\psi_y(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} B_{mn}\sin(m\pi x/a)\ \cos(n\pi y/b)$$

$$w(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn}\sin(m\pi x/a)\ \sin(n\pi y/b)$$

$$u_o(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} E_{mn}\cos(m\pi x/a)\ \sin(n\pi y/b)$$

$$v_o(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} G_{mn}\sin(m\pi x/a)\ \cos(n\pi y/b) \qquad (2.50)$$

The single terms associated with the variations of the degrees of freedom are:

$$\delta u_o \longrightarrow \cos(p\pi x/a)\ \sin(q\pi y/b)$$

$$\delta v_o \longrightarrow \sin(p\pi x/a)\ \cos(q\pi y/b)$$

$$\delta w \longrightarrow \sin(p\pi x/a)\ \sin(q\pi y/b)$$

$$\delta\psi_x \longrightarrow \cos(p\pi x/a)\ \sin(q\pi y/b)$$

$$\delta\psi_y \longrightarrow \sin(p\pi x/a)\ \cos(q\pi y/b) \qquad (2.51)$$

Notice the indices in Eq (2.51) are p and q. As explained in the last section, the single terms that replace the variations of the degrees of freedom govern the number of Galerkin equations. Therefore, the number of terms in each equation is governed by m and n, and the number of equations is governed by p and q. (See Appendix E.)

This thesis would not have been possible without the use of a symbolic manipulation program such as MACSYMA (25). This

artificial intelligence based program proved to be invaluable in the algebraic manipulation and integration of the equations of motion and boundary conditions in Eqs (2.39) through (2.43).

The step-by-step process taken to utilize MACSYMA for generating the Galerkin equations for Eqs (2.39) through (2.43) is outlined below. (Note at this point $\bar{N}_2$ and $\bar{N}_6$ are set equal to zero, since only axial buckling is considered.)

1. Substitute Eq (2.50) into the five equations and evaluate the appropriate derivatives. For example, in Eq (2.39) there are terms such as $u_{o,yy}$, $w_{,yyy}$, and $\psi_{x,xy}$ that need to be evaluated.

2. Substitute the single term expressions for δu_o, δv_o, δw, $\delta\psi_x$, and $\delta\psi_y$ from Eq (2.51) into the five equations.

3. Integrate all five equations according to guidelines outlined in Appendix C. For each equation this includes a double integration for the equation of motion and two single integrations for the edge boundary conditions. The results of the integration depend directly upon the values of m, n, p, and q; there are nonzero results for only two cases:

Case (1): $m = p$ and $n = q$

Case (2): $m \neq p$, $(m + p)$ odd and $n \neq q$, $(n + q)$ odd.

If m,n,p, and q do not meet the criterion of these two cases, the five equations become equal to zero when integrated.

4. Collect terms and simplify the equations.

The generated Galerkin equations for case (1) are shown below.

Equation (2.39) for u_o becomes:

$$\left\{\pi^2 aq^2(3h^2D_{66} - 4F_{66})/(8bh^2R)\right\}A_{mn} +$$

$$\left\{\pi^2 pq(3h^2D_{66} + 6h^2D_{12} - 4F_{66} - 8F_{12})/(24h^2R)\right\}B_{mn} -$$

$$\left\{\left[\pi^3 pq^2(8F_{66} + 4F_{12}) - 3\pi b^2h^2pA_{12}\right]/(12bh^2R)\right\}C_{mn} -$$

$$\left\{\pi^2\left[4R^2(a^2q^2A_{66} + b^2p^2A_{11}) + a^2q^2D_{66}\right]/(16abR^2)\right\}E_{mn} -$$

$$\left\{\pi^2 pq\left[4R^2(A_{66} + A_{12}) - D_{66}\right]/(16R^2)\right\}G_{mn} = 0 \qquad (2.52)$$

Equation (2.40) for v_o becomes:

$$\left\{\pi^2 pq(3h^2D_{66} - 4F_{66})/(24h^2R)\right\}A_{mn} +$$

$$\left\{\pi^2\left[a^2q^2(6h^2D_{22} - 8F_{22}) + b^2p^2(4F_{66} - 3h^2D_{66})\right]/(24abh^2R)\right\}B_{mn}$$

$$- \left\{\pi aq(4\pi^2q^2F_{22} - 3b^2h^2A_{22})/(12b^2h^2R)\right\}C_{mn} -$$

$$\left\{\pi^2 pq\left[4R^2(A_{66} + A_{12}) - D_{66}\right]/(16R^2)\right\}E_{mn} -$$

$$\left\{\pi^2\left[4R^2(a^2q^2A_{22} + b^2p^2A_{66}) + b^2p^2D_{66}\right]/(16abR^2)\right\}G_{mn} =$$

$$- \left\{ab\bar{I}_2'/4\right\}\omega^2B_{mn} + \left\{\pi aq\bar{I}_3'/4\right\}\omega^2C_{mn} \qquad (2.53)$$

Equation (2.41) for w becomes:

$$
\Big\{-\pi p\Big[R^2\Big(4\pi^2a^2q^2(8H_{66} + 4H_{12} - 6h^2F_{66} - 3h^2F_{12})
$$

$$
+ 4\pi^2b^2p^2(4H_{11} - 3h^2F_{11}) + 9a^2b^2(h^4A_{55} - 8h^2D_{55} + 16F_{55})\Big)
$$

$$
+ 4\pi^2a^2q^2(4J_{66} - 3h^2H_{66})\Big]/(36a^2bh^4R^2)\Big\}A_{mn} -
$$

$$
\Big\{\pi q\Big[R^2\Big(4\pi^2a^2q^2(4H_{22} - 3h^2F_{22}) + 4\pi^2b^2p^2(8H_{66} + 4H_{12} - 6h^2F_{66}
$$

$$
- 3h^2F_{12}) + 9a^2b^2(h^4A_{44} - 8h^2D_{44} + 16F_{44})\Big) + 4\pi^2a^2q^2(4J_{22} -
$$

$$
3h^2H_{22}) + a^2b^2h^2(9h^2D_{22} - 12F_{22})\Big]/(36ab^2h^4R^2)\Big\}B_{mn} -
$$

$$
\Big\{\Big[\pi^2R^2\Big(16\pi^2a^4q^4H_{22} + a^2b^2q^2(64\pi^2p^2H_{66} + 32\pi^2p^2H_{12} + 9a^2h^4A_{44} -
$$

$$
72a^2h^2D_{44} + 144a^2F_{44}) + 16\pi^2b^4p^4H_{11} + 9a^2b^4p^2(h^4A_{55} - 8h^2D_{55}
$$

$$
+ 16F_{55})\Big) + 16\pi^4a^4q^4J_{22} + 8\pi^2a^2b^2q^2(2\pi^2p^2J_{66} - 3a^2h^2F_{22}) +
$$

$$
9a^4b^4h^4A_{22}\Big]/(36a^3b^3h^4R^2)\Big\}C_{mn} -
$$

$$
\Big\{\pi p(8\pi^2q^2F_{66} + 4\pi^2q^2F_{12} - 3b^2h^2A_{12})/(12bh^2R)\Big\}E_{mn} -
$$

$$
\Big\{\pi aq(4\pi^2q^2F_{22} - 3b^2h^2A_{22})/(12b^2h^2R)\Big\}G_{mn} =
$$

$$
\Big\{\pi bp\bar{I}_5/4\Big\}\omega^2A_{mn} + \Big\{\pi aq\bar{I}_5/4\Big\}\omega^2B_{mn} - \Big\{\Big[16\pi^2(a^2q^2 + b^2p^2)I_7 +
$$

$$
9a^2b^2h^4I_1\Big]/(36abh^4)\Big\}\omega^2C_{mn} + \Big\{\pi^2bp^2/(4a)\Big\}\bar{N}_1C_{mn} \qquad (2.54)
$$

Equation (2.42) for ψ_x becomes:

$$\left\{-\left[R^2\left\{\pi^2a^2q^2(16H_{66} + 9h^4D_{66} - 24h^2F_{66}) + \right.\right.\right.$$

$$\pi^2b^2p^2(16H_{11} + 9h^4D_{11} - 24h^2F_{11}) + 9a^2b^2(h^4A_{55} - 8h^2D_{55} +$$

$$\left.\left.\left.16F_{55})\right\} + \pi^2a^2q^2(16J_{66} - 24h^2H_{66} + 9h^4F_{66})\right]/(36abh^4R^2)\right\}A_{mn} -$$

$$\left\{\pi^2pq\left[16(H_{66}+ H_{12})+ 9h^4(D_{66}+D_{12})-24h^2(F_{66} + F_{12})\right]/(36h^4)\right\}B_{mn}$$

$$- \left\{\pi p\left[R^2\left\{4\pi^2a^2q^2(8H_{66} + 4H_{12} - 6h^2F_{66} - 3h^2F_{12}) + 4\pi^2b^2p^2(4H_{11}\right.\right.\right.$$

$$\left.- 3h^2F_{11}) + 9a^2b^2(h^4A_{55} - 8h^2D_{55} + 16F_{55})\right\} + 4\pi^2a^2q^2(4J_{66} -$$

$$\left.\left.3h^2H_{66})\right]/(36a^2bh^4R^2)\right\}C_{mn} +$$

$$\left\{\pi^2aq^2(3h^2D_{66} - 4F_{66})/(8bh^2R)\right\}E_{mn} +$$

$$\left\{\pi^2pq(3h^2D_{66} - 4F_{66})/(24h^2R)\right\}G_{mn} =$$

$$\left\{-ab\bar{I}_4/4\right\}\omega^2A_{mn} + \left\{\pi bp\bar{I}_5/4\right\}\omega^2C_{mn} \tag{2.55}$$

And finally, Eqn (2.43) for ψ_y becomes:

$$\Big\{-\pi^2 pq\Big[16(H_{66} + H_{12}) + 9h^4(D_{66} + D_{12}) -$$

$$24h^2(F_{66} + F_{12})\Big]/(36h^4)\Big\}A_{mn} -$$

$$\Big\{\Big[R^2\Big(\pi^2 a^2 q^2(16H_{22} + 9h^4 D_{22} - 24h^2 F_{22}) + \pi^2 b^2 p^2(16H_{66} +$$

$$9h^4 D_{66} - 24h^2 F_{66}) + 9a^2 b^2(h^4 A_{44} - 8h^2 D_{44} + 16F_{44})\Big) +$$

$$\pi^2 a^2 q^2(16J_{22} - 24h^2 H_{22} + 9h^4 F_{22})\Big]/(36abh^4 R^2)\Big\}B_{mn} -$$

$$\Big\{\pi q\Big[R^2\Big(4\pi^2 a^2 q^2(4H_{22} - 3h^2 F_{22}) + 4\pi^2 b^2 p^2(8H_{66} + 4H_{12} - 6h^2 F_{66}$$

$$- 3h^2 F_{12}) + 9a^2 b^2(h^4 A_{44} - 8h^2 D_{44} + 16F_{44})\Big) + 4\pi^2 a^2 q^2(4J_{22} -$$

$$3h^2 H_{22}) + 3a^2 b^2 h^2(3h^2 D_{22} - 4F_{22})\Big]/(36ab^2 h^4 R^2)\Big\}C_{mn} +$$

$$\Big\{\pi^2 pq\Big[3h^2(D_{66} + 2D_{12}) - 4(F_{66} + 2F_{12})\Big]/(24h^2 R)\Big\}E_{mn} +$$

$$\Big\{\pi^2\Big[a^2 q^2(6h^2 D_{22} - 8F_{22}) + b^2 p^2(4F_{66} - 3h^2 D_{66})\Big]/(24abh^2 R)\Big\}G_{mn}$$

$$= \Big\{-ab\bar{I}_4/4\Big\}\omega^2 B_{mn} + \Big\{\pi aq\bar{I}_5/4\Big\}\omega^2 C_{mn} \qquad (2.56)$$

The set of Galerkin equations for Case (2) are shown below. That is, when $m \neq p$, $(m + p)$ odd and $n \neq q$, $(n + q)$ odd, the following set of Galerkin equations are obtained when Eqs (2.39) to (2.43) are integrated:

Equation (2.39) for u_o becomes:

$$\left\{-2nq\left[p^2(6h^2D_{16} - 8F_{16}) + m^2(3h^2D_{16} - 4F_{16})\right]/[3h^2R(p^2 - m^2)(q^2 - n^2)]\right\}A_{mn} -$$
$$\left\{2amn^2q(3h^2D_{26} - 4F_{26})/[bh^2R(p^2 - m^2)(q^2 - n^2)]\right\}B_{mn} +$$
$$\left\{4mnq\left[\pi^2(4b^2p^2F_{16} + 6a^2n^2F_{26}) + b^2(8\pi^2m^2F_{16} - 12a^2h^2A_{26})\right]/[3\pi ab^2h^2R(p^2 - m^2)(q^2 - n^2)]\right\}C_{mn} +$$
$$\left\{4nq(p^2 + m^2)A_{16}/[(p^2 - m^2)(q^2 - n^2)]\right\}E_{mn} +$$
$$\left\{4mq(b^2p^2A_{16} + a^2n^2A_{26})/[ab(p^2 - m^2)(q^2 - n^2)]\right\}G_{mn} = 0 \tag{2.57}$$

From Equation (2.40) for v_o becomes:

$$\left\{-np\left[4a^2q^2(3h^2D_{26} - 4F_{26}) + 2b^2m^2(4F_{16} - 3h^2D_{16})\right]/[3abh^2R(p^2 - m^2)(q^2 - n^2)]\right\}A_{mn} -$$
$$\left\{mn^2p(6h^2D_{26} - 8F_{26})/[3h^2R(p^2 - m^2)(q^2 - n^2)]\right\}B_{mn} +$$
$$\left\{4mnp\left[2\pi^2a^2(2q^2 + n^2)F_{26} - 2\pi^2b^2m^2F_{16} - 3a^2b^2h^2A_{26}\right]/[3\pi a^2bh^2R(p^2 - m^2)(q^2 - n^2)]\right\}C_{mn} +$$
$$\left\{4np(a^2q^2A_{26} + b^2m^2A_{16})/[ab(p^2 - m^2)(q^2 - n^2)]\right\}E_{mn} +$$
$$\left\{4mp(q^2 + n^2)A_{26}/[(p^2 - m^2)(q^2 - n^2)]\right\}G_{mn} = 0 \tag{2.58}$$

Equation (2.41) for w becomes:

$$\Big\{pqn\Big[R^2\Big(16\pi^2a^2n^2(4H_{26} - 3h^2F_{26}) + 48\pi^2b^2m^2(4H_{16} -$$

$$3h^2F_{16}) + 36a^2b^2(h^4A_{45} - 8h^2D_{45} + 16F_{45})\Big) + 4a^2\Big(4\pi^2n^2(4J_{26} -$$

$$3h^2H_{26}) + 3b^2h^2(3h^2D_{26} - 4F_{26})\Big)\Big]/[9\pi ab^2h^4R^2(p^2-m^2)(q^2-n^2)]\Big\}A_{mn}$$

$$+ \Big\{mpq\Big[R^2\Big(48\pi^2a^2n^2(4H_{26} - 3h^2F_{26}) + 16\pi^2b^2m^2(4H_{16} - 3h^2F_{16}) +$$

$$36a^2b^2(h^4A_{45} - 8h^2D_{45} + 16F_{45})\Big) +$$

$$16\pi^2a^2n^2(4J_{26} - 3h^2H_{26})\Big]/[9\pi a^2bh^4R^2(p^2 - m^2)(q^2 - n^2)]\Big\}B_{mn} +$$

$$\Big\{mnpq\Big[R^2\Big(256\pi^2(a^2n^2H_{26} + b^2m^2H_{16}) + 72a^2b^2(h^4A_{45} - 8h^2D_{45} +$$

$$16F_{45})\Big) + 32a^2(4\pi^2n^2J_{26} -$$

$$3b^2h^2F_{26})\Big]/[9a^2b^2h^4R^2(p^2 - m^2)(q^2 - n^2)]\Big\}C_{mn} +$$

$$\Big\{4npq(2\pi^2a^2n^2F_{26} + 2\pi^2b^2m^2F_{16} -$$

$$a^2b^2h^2A_{26})/[\pi ab^2h^2R(p^2 - m^2)(q^2 - n^2)]\Big\}E_{mn} +$$

$$\Big\{4mpq(6\pi^2a^2n^2F_{26} - 2\pi^2b^2m^2F_{16} -$$

$$3a^2b^2h^2A_{26})/[3\pi a^2bh^2R(p^2 - m^2)(q^2 - n^2)]\Big\}G_{mn} = 0 \qquad (2.59)$$

Equation (2.42) for ψ_x becomes:

$$\Big\{4nq\Big[32p^2H_{16} + 9h^4(p^2 + m^2)D_{16} - 12h^2(3p^2 +$$

$$m^2)F_{16}\Big]/[9h^4(p^2 - m^2)(q^2 - n^2)]\Big\}A_{mn} +$$

$$\Big\{mq\Big[4R^2\Big[\pi^2b^2p^2(32H_{16} + 9h^4D_{16} - 36h^2F_{16}) + \pi^2a^2n^2(16H_{26} +$$

$$9h^4D_{26} - 24h^2F_{26}) + 4\pi^2b^2m^2(3h^2F_{16} - 4H_{16}) + 9a^2b^2(h^4A_{45} -$$

$$8h^2D_{45} + 36F_{45})\Big] + 4\pi^2a^2n^2(16J_{26} - 24h^2H_{26} +$$

$$9h^4F_{26})\Big]/[9\pi^2abh^4R^2(p^2 - m^2)(q^2 - n^2)]\Big\}B_{mn} +$$

$$\Big\{mnq\Big[4R^2\Big[8\pi^2b^2p^2(8H_{16} - 3h^2F_{16}) + 4\pi^2a^2n^2(4H_{26} - 3h^2F_{26}) -$$

$$4\pi^2b^2m^2(4H_{16} + 3h^2F_{16}) + 9a^2b^2(h^4A_{45} - 8h^2D_{45} + 16F_{45})\Big] +$$

$$4a^2\Big[4\pi^2n^2(4J_{26} - 3h^2H_{26}) +$$

$$3a^2b^2h^2(3h^2D_{26} - 4F_{26})\Big]\Big]/[9\pi ab^2h^4R^2(p^2 - m^2)(q^2 - n^2)]\Big\}C_{mn} -$$

$$\Big\{2nq\Big[3h^2(p^2 + 2m^2)D_{16} - 4(2p^2 +$$

$$m^2)F_{16}\Big]/[3h^2R(p^2 - m^2)(q^2 - n^2)]\Big\}E_{mn} +$$

$$\Big\{2mq\Big[3b^2h^2p^2D_{16} + 4b^2(m^2 - 2p^2)F_{16} + 2a^2n^2(4F_{26} -$$

$$3h^2D_{26})\Big]/[3abh^2R(p^2 - m^2)(q^2 - n^2)]\Big\}G_{mn} = 0 \tag{2.60}$$

And finally, Eq (2.43) for ψ_y becomes:

$$\Big\{np\Big[R^2\Big(4\pi^2a^2[16(2q^2 - n^2)H_{26} + 9h^4q^2D_{26} -$$

$$12h^2(3q^2 - n^2)F_{26}] + 4\pi^2b^2m^2(16H_{16} + 9h^4D_{16} - 24h^2F_{16}) +$$

$$36a^2b^2(h^4A_{45} - 8h^2D_{45} + 16F_{45})\Big) + 4\pi^2a^2\Big(16(2q^2 - n^2)J_{26} -$$

$$12h^2(3q^2 - n^2)H_{26} + 9h^4q^2F_{26}\Big)\Big]/[9\pi^2abh^4R^2(p^2-m^2)(q^2-n^2)]\Big\}A_{mn}$$

$$+ \Big\{4mp\Big[32q^2H_{26} + 9h^4(q^2 + n^2)D_{26} -$$

$$12h^2(3q^2 + p^2)F_{26}\Big]/[9h^4(p^2 - m^2)(q^2 - n^2)]\Big\}B_{mn} +$$

$$\Big\{mnp\Big[4R^2\Big(4\pi^2a^2[4(4q^2-n^2)H_{26}- 3h^2(2q^2+ n^2)F_{26}] + 4\pi^2b^2m^2(4H_{16}$$

$$- 3h^2F_{16}) + 36a^2b^2(h^4A_{45} - 8h^2D_{45} + 16F_{45})\Big) + 16\pi^2a^2\Big(4(2q^2 -$$

$$n^2)J_{26} - 3h^2q^2H_{26}\Big)\Big]/[9\pi a^2bh^4R^2(p^2 - m^2)(q^2 - n^2)]\Big\}C_{mn} -$$

$$\Big\{2anp\Big[3q^2h^2D_{26} - 4(2q^2- n^2)F_{26}\Big]/[bh^2R(p^2- m^2)(q^2- n^2)]\Big\}E_{mn} -$$

$$\Big\{2mp\Big[3q^2h^2D_{26}- 4(2q^2- n^2)F_{26}\Big]/[3h^2R(p^2- m^2)(q^2- n^2)]\Big\}G_{mn} = 0 \tag{2.61}$$

Equations (2.52) to (2.61) are now ready to be put in matrix format and then input into the eigenvalue subroutine to solve for either ω^2 or $\bar{N}_1$. The details of this procedure are explained in the next chapter.

CLAMPED BOUNDARY CONDITION

In this section the admissible functions that satisfy clamped boundary conditions on all four edges of the circular cylindrical shell panel are chosen. The following bending boundary conditions exist:

At $x = 0$ and $x = a$

$$w = \psi_x = \psi_y = 0$$

and

At $y = 0$ and $y = b$

$$w = \psi_x = \psi_y = 0$$

The membrane boundary conditions will be the same as those in the previous section: from Jones (9), a C-2 type boundary condition.

At $x = 0$ and $x = a$

$$u_o \neq 0 \text{ and } v_o = 0$$

and

At $y = 0$ and $y = b$

$$u_o = 0 \text{ and } v_o \neq 0$$

The admissible functions in Eq (2.49) become:

$$\psi_x(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} \sin(m\pi x/a) \sin(n\pi y/b)$$

$$\psi_y(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} B_{mn} \sin(m\pi x/a) \sin(n\pi y/b)$$

$$w(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn} \sin(m\pi x/a) \sin(n\pi y/b)$$

$$u_o(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} E_{mn} \cos(m\pi x/a) \sin(n\pi y/b)$$

$$v_o(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} G_{mn} \sin(m\pi x/a) \cos(n\pi y/b) \qquad (2.62)$$

The single terms associated with the variations of the degrees of freedom are:

$$\begin{aligned}
\delta u_o &\rightarrow \sin(p\pi x/a) \sin(q\pi y/b) \\
\delta v_o &\rightarrow \sin(p\pi x/a) \sin(q\pi y/b) \\
\delta w &\rightarrow \sin(p\pi x/a) \sin(q\pi y/b) \\
\delta \psi_x &\rightarrow \cos(p\pi x/a) \sin(q\pi y/b) \\
\delta \psi_y &\rightarrow \sin(p\pi x/a) \cos(q\pi y/b) \qquad (2.63)
\end{aligned}$$

And, as in the simply supported case, the indices for the single terms are p and q; m and n govern the number of terms per equation, and p and q govern the number of equations.

The procedure for generating the Galerkin equations is exactly the same as in the last section. However, as outlined in Appendix C, after integrating Eqs (2.39) to (2.43), four cases give nonzero results:

Case (1): $m = p$ and $n = q$

Case (2): $m = p$ and $n \neq q$, $(n + q)$ odd

Case (3): $m \neq p$, $(m + p)$ odd and $n = q$

Case (4): $m \neq p$, $(m + p)$ odd and $n \neq q$, $(n + q)$ odd

If m,n,p, and q do not meet the criterion of these four cases, the five equations become equal to zero when integrated.

The generated Galerkin equations for case (1) are shown below.

Equation (2.39) for u_o becomes:

$$0 \cdot A_{mn} + 0 \cdot B_{mn} -$$

$$\left\{\pi p\left[\pi^2 q^2 (8F_{66} + 4F_{12}) - 3b^2h^2A_{12}\right]/(12bh^2R)\right\}C_{mn} -$$

$$\left\{\pi^2\left[4R^2(a^2q^2A_{66} + b^2p^2A_{11}) + a^2q^2D_{66}\right]/(16abR^2)\right\}E_{mn} -$$

$$\left\{\pi^2 qp\left[4R^2(A_{66} + A_{12}) - D_{66}\right]/(16R^2)\right\}G_{mn} = 0 \qquad (2.64)$$

Equation (2.40) for v_o becomes:

$$0 \cdot A_{mn} + 0 \cdot B_{mn} -$$

$$\left\{\pi aq(4\pi^2q^2F_{22} - 3b^2h^2A_{22})/(12b^2h^2R)\right\}C_{mn} -$$

$$\left\{\pi^2 pq\left[4R^2(A_{66} + A_{12}) - D_{66}\right]/(16R^2)\right\}E_{mn} -$$

$$\left\{\pi^2\left[4R^2(a^2q^2A_{22} + b^2p^2A_{66}) + b^2p^2D_{66}\right]/(16abR^2)\right\}G_{mn} =$$

$$\left\{\pi aq\bar{I}_3{}'/4\right\}\omega^2 C_{mn} \qquad (2.65)$$

Equation (2.41) for w becomes:

$$
\begin{aligned}
&0 \cdot A_{mn} + 0 \cdot B_{mn} - \\
&\Big\{\Big[\pi^2R^2\Big[32\pi^2a^2b^2p^2q^2(2H_{66} + H_{12}) + 9a^4b^2q^2(h^4A_{44} - 8h^2D_{44} + \\
&16F_{44}) + 16\pi^2(a^4q^4H_{22} + b^4p^4H_{11}) + 9a^2b^4p^2(h^4A_{55} - 8h^2D_{55} + \\
&16F_{55})\Big] + 16\pi^4a^4q^4J_{22} + 8\pi^2a^2b^2q^2(2p^2\pi^2J_{66} - 3a^2h^2F_{22}) + \\
&9a^4b^4h^4A_{22}\Big]/(36a^3b^3h^4R^2)\Big\}C_{mn} - \\
&\Big\{\pi p\Big[4\pi^2q^2(2F_{66} + F_{12}) - 3b^2h^2A_{12}\Big]/(12bh^2R)\Big\}E_{mn} - \\
&\Big\{\pi aq(4\pi^2q^2F_{22} - 3b^2h^2A_{22})/(12b^2h^2R)\Big\}G_{mn} = \\
&\Big\{-\Big[16\pi^2(a^2q^2 + b^2p^2)I_7 + 9a^2b^2h^4I_1\Big]/(36abh^4)\Big\}\omega^2C_{mn} + \\
&\Big\{\pi^2bp^2/(4a)\Big\}\bar{N}_1C_{mn}
\end{aligned}
\qquad (2.66)
$$

Equation (2.42) for ψ_x becomes:

$$\Big\{-\Big[R^2\Big[\pi^2a^2q^2(16H_{66} + 9h^4D_{66} - 24h^2F_{66}) +$$

$$\pi^2b^2p^2(16H_{11} + 9h^4D_{11} - 24h^2F_{11}) + 9a^2b^2(h^4A_{55} - 8h^2D_{55} +$$

$$16F_{55})\Big] + \pi^2a^2q^2(16J_{66} - 24h^2H_{66} + 9h^4F_{66})\Big]/(36abh^4R^2)\Big\}A_{mn} -$$

$$\Big\{\Big[R^2\Big[\pi^2a^2q^2(16H_{26} + 9h^4D_{26} - 24h^2F_{26}) + \pi^2b^2p^2(16H_{16} +$$

$$9h^4D_{16} - 24h^2F_{16}) + 9a^2b^2(h^4A_{45} - 8h^2D_{45} + 16F_{45})\Big] +$$

$$\pi^2a^2q^2(16J_{26} - 24h^2H_{26} + 9h^4F_{26})\Big]/(36abh^4R^2)\Big\}B_{mn} +$$

$$0 \cdot C_{mn} + 0 \cdot E_{mn} + 0 \cdot G_{mn} = \Big\{-ab\bar{I}_4/4\Big\}\omega^2A_{mn} \qquad (2.67)$$

And finally, Eq (2.43) for ψ_y becomes:

$$\Big\{-\Big[R^2\Big[\pi^2a^2q^2(16H_{26} + 9h^4D_{26} - 24h^2F_{26}) +$$

$$\pi^2b^2p^2(16H_{16} + 9h^4D_{16} - 24h^2F_{16}) + 9a^2b^2(h^4A_{45} - 8h^2D_{45} +$$

$$16F_{45})\Big] + \pi^2a^2q^2(16J_{26} - 24h^2H_{26} + 9h^4F_{26})\Big]/(36abh^4R^2)\Big\}A_{mn} -$$

$$\Big\{\Big[R^2\Big[\pi^2a^2q^2(16H_{22} + 9h^4D_{22} - 24h^2F_{22}) + \pi^2b^2p^2(16H_{66} +$$

$$9h^4D_{66} - 24h^2F_{66}) + 9a^2b^2(h^4A_{44} - 8h^2D_{44} + 16F_{44})\Big] +$$

$$\pi^2a^2q^2(16J_{22} - 24h^2H_{22} + 9h^4F_{22})\Big]/(36abh^4R^2)\Big\}B_{mn} +$$

$$0 \cdot C_{mn} + 0 \cdot E_{mn} + 0 \cdot G_{mn} = \Big\{-ab\bar{I}_4/4\Big\}\omega^2B_{mn} \qquad (2.68)$$

The set of Galerkin equations for Case (2) are shown below. That is, when $m = p$ and $n \neq q$, $(n + q)$ odd, the following set of equations are obtained when Eqs (2.39) to (2.43) are integrated. Equation (2.39) for u_o becomes:

$$\left\{-\pi npq(3h^2D_{16} - 4F_{16})/[2h^2R(q^2 - n^2)]\right\}A_{mn} -$$

$$\left\{\pi npq\left[3h^2(D_{66} + 2D_{12}) - 4(F_{66} + 2F_{12})\right]/[6h^2R(q^2 - n^2)]\right\}B_{mn} +$$

$$0 \cdot C_{mn} + 0 \cdot E_{mn} + 0 \cdot G_{mn} = 0 \qquad (2.69)$$

Equation (2.40) for v_o becomes:

$$\left\{-n\pi\left[2a^2q^2(3h^2D_{26} - 4F_{26}) + b^2p^2(4F_{16} -\right.\right.$$

$$\left.\left.3h^2D_{16})\right]/[6abh^2R(q^2 - n^2)]\right\}A_{mn} -$$

$$\left\{ n\pi\left[2a^2q^2(3h^2D_{22} - 4F_{22}) + b^2p^2(4F_{66} -\right.\right.$$

$$\left.\left.3h^2D_{66})\right]/[6abh^2R(q^2 - n^2)]\right\}B_{mn} +$$

$$0 \cdot C_{mn} + 0 \cdot E_{mn} + 0 \cdot G_{mn} = \left\{abn\bar{I}_2{}'/[\pi(q^2 - n^2)]\right\}\omega^2B_{mn}$$

$$(2.70)$$

Equation (2.41) for w becomes:

$$\Big\{nq\Big[R^2\Big(12\pi^2b^2p^2(4H_{16} - 3h^2F_{16}) + 4\pi^2a^2n^2(4H_{26} -$$

$$3h^2F_{26}) + 9a^2b^2(h^4A_{45} - 8h^2D_{45} + 16F_{45})\Big) + a^2\Big(4\pi^2n^2(4J_{26} -$$

$$3h^2H_{26}) + 3b^2h^2(3h^2D_{26} - 4F_{26})\Big)\Big]/[9ab^2h^4R^2(q^2 - n^2)]\Big\}A_{mn} +$$

$$\Big\{qn\Big[R^2\Big(4\pi^2b^2p^2(8H_{66} + 4H_{16} - 6h^2F_{66} - 3h^2F_{12}) + 4\pi^2a^2n^2(4H_{22}$$

$$- 3h^2F_{22}) + 9a^2b^2(h^4A_{44} - 8h^2D_{44} + 16F_{44})\Big) + a^2\Big(4\pi^2n^2(4J_{22} -$$

$$3h^2H_{22}) + 3b^2h^2(3h^2D_{22} - 4F_{22})\Big)\Big]/[9ab^2h^4R^2(q^2 - n^2)]\Big\}B_{mn} +$$

$$0 \cdot C_{mn} + 0 \cdot E_{mn} + 0 \cdot G_{mn} = -\Big\{anq\bar{I}_5/(q^2 - n^2)\Big\}\omega^2B_{mn} \tag{2.71}$$

Equation (2.42) for ψ_x becomes:

$$0 \cdot A_{mn} + 0 \cdot B_{mn} -$$

$$\Big\{qn\Big[R^2\Big(12\pi^2b^2p^2(4H_{16} - 3h^2F_{16}) + 4\pi^2n^2a^2(4H_{26} - 3h^2F_{26}) +$$

$$9a^2b^2(h^4A_{45} - 8h^2D_{45} + 16F_{45})\Big) + a^2\Big(4\pi^2n^2(4J_{26} - 3h^2H_{26}) +$$

$$3b^2h^2(3h^2D_{26} - 4F_{26})\Big)\Big]/[9ab^2h^4R^2(q^2 - n^2)]\Big\}C_{mn} +$$

$$\Big\{\pi npq(3h^2D_{16} - 4F_{16})/[2h^2R(q^2 - n^2)]\Big\}E_{mn} -$$

$$\Big\{\pi q\Big[b^2p^2(3h^2D_{16} - 4F_{16}) + 2a^2n^2(4F_{26} -$$

$$3h^2D_{26})\Big]/[6abh^2R(q^2 - n^2)]\Big\}G_{mn} = 0 \tag{2.72}$$

And finally, Eq (2.43) for ψ_y becomes:

$$0 \cdot A_{mn} + 0 \cdot B_{mn} -$$

$$\Big\{qn\Big[R^2\Big[4\pi^2b^2p^2(8H_{66} + 4H_{12} - 6h^2F_{66} - 3h^2F_{12}) + 4\pi^2n^2a^2(4H_{22}$$

$$- 3h^2F_{22}) + 9a^2b^2(h^4A_{44} - 8h^2D_{44} + 16F_{44})\Big] + a^2\Big[4\pi^2n^2(4J_{22} -$$

$$3h^2H_{22}) + 3b^2h^2(3h^2D_{22} - 4F_{22})\Big]\Big]/[9ab^2h^4R^2(q^2 - n^2)]\Big\}C_{mn} +$$

$$\Big\{\pi npq\Big[3h^2(D_{66} + 2D_{12}) - 4(F_{66} + 2F_{12})\Big]/[6h^2R(q^2 - n^2)]\Big\}E_{mn} -$$

$$\Big\{\pi q\Big[b^2p^2(3h^2D_{66} - 4F_{66}) + 2a^2n^2(4F_{22} -$$

$$3h^2D_{22})\Big]/[6abh^2R(q^2 - n^2)]\Big\}G_{mn} = \Big\{anq\bar{I}_5/(q^2 - n^2)\Big\}\omega^2C_{mn} \tag{2.73}$$

The set of Galerkin equations for Case (3) in which $m \neq p$, $(m+p)$ odd and $n = q$ are shown below.

Equation (2.39) for u_o becomes:

$$\Big\{\pi maq^2(3h^2D_{66} - 4F_{66})/[2bh^2R(p^2 - m^2)]\Big\}A_{mn} -$$

$$\Big\{\pi maq^2(3h^2D_{26} - 4F_{26})/[2bh^2R(p^2 - m^2)]\Big\}B_{mn} +$$

$$0 \cdot C_{mn} + 0 \cdot E_{mn} + 0 \cdot G_{mn} = 0 \tag{2.74}$$

Equation (2.40) for v_o becomes:

$$\left\{-\pi pqm(3h^2D_{66} - 4F_{66})/[6h^2R(p^2 - m^2)]\right\}A_{mn} -$$

$$\left\{\pi pqm(3h^2D_{26} - 4F_{26})/[6h^2R(p^2 - m^2)]\right\}B_{mn} +$$

$$0 \cdot C_{mn} + 0 \cdot E_{mn} + 0 \cdot G_{mn} = 0 \tag{2.75}$$

Equation (2.41) for w becomes:

$$\left\{mp\left[R^2\left(4\pi^2a^2q^2(8H_{66} + 4H_{12} - 6h^2F_{66} - 3h^2F_{12}) +\right.\right.\right.$$

$$\left.4\pi^2b^2m^2(4H_{11} - 3h^2F_{11}) + 9a^2b^2(h^4A_{55} - 8h^2D_{55} + 16F_{55})\right) +$$

$$\left.\left.4\pi^2a^2q^2(4J_{66} - 3h^2H_{66})\right]/[9a^2bh^4R^2(p^2 - m^2)]\right\}A_{mn} +$$

$$\left\{mp\left[R^2\left(12\pi^2a^2q^2(4H_{26} - 3h^2F_{26}) + 4\pi^2b^2m^2(4H_{16} - 3h^2F_{16}) +\right.\right.\right.$$

$$\left.9a^2b^2(h^4A_{45} - 8h^2D_{45} + 16F_{45})\right) + 4\pi^2a^2q^2(4J_{26} -$$

$$\left.\left.3h^2H_{26})\right]/[9a^2bh^4R^2(p^2 - m^2)]\right\}B_{mn} +$$

$$0 \cdot C_{mn} + 0 \cdot E_{mn} + 0 \cdot G_{mn} = -\left\{bmp\overline{I}_5/(p^2 - m^2)\right\}\omega^2A_{mn} \tag{2.76}$$

Equation (2.42) for ψ_x becomes:

$$0 \cdot A_{mn} + 0 \cdot B_{mn} -$$

$$\Big\{mp\Big[R^2\Big(4\pi^2a^2q^2(8H_{66} + 4H_{12} - 6h^2F_{66} - 3h^2F_{12}) + 4\pi^2b^2m^2(4H_{11}$$

$$- 3h^2F_{11}) + 9a^2b^2(h^4A_{55} - 8h^2D_{55} + 16F_{55})\Big) + 4\pi^2a^2q^2(4J_{66} -$$

$$3h^2H_{66})\Big]/[9a^2bh^4R^2(p^2 - m^2)]\Big\}C_{mn} +$$

$$\Big\{\pi apq^2(3h^2D_{66} - 4F_{66})/[2bh^2R(p^2 - m^2)]\Big\}E_{mn} +$$

$$\Big\{\pi mpq(3h^2D_{66} - 4F_{66})/[6h^2R(p^2 - m^2)]\Big\}G_{mn} = \Big\{bmp\bar{I}_5/(p^2 - m^2)\Big\}\omega^2C_{mn} \tag{2.77}$$

And finally, Eq (2.43) for ψ_y becomes:

$$0 \cdot A_{mn} + 0 \cdot B_{mn} -$$

$$\Big\{mp\Big[R^2\Big(12\pi^2a^2q^2(4H_{26} - 3h^2F_{26}) + 4\pi^2b^2m^2(4H_{16} - 3h^2F_{16}) +$$

$$9a^2b^2(h^4A_{45} - 8h^2D_{45} + 16F_{45})\Big) + 4\pi^2a^2q^2(4J_{26} -$$

$$3h^2H_{26})\Big]/[9a^2bh^4R^2(p^2 - m^2)]\Big\}C_{mn} +$$

$$\Big\{\pi apq^2(3h^2D_{26} - 4F_{26})/[2bh^2R(p^2 - m^2)]\Big\}E_{mn} +$$

$$\Big\{\pi mpq(3h^2D_{26} - 4F_{26})/[6h^2R(p^2 - m^2)]\Big\}G_{mn} = 0 \tag{2.78}$$

The last set of Galerkin equations is for Case (4) in which $m \neq p$, $(m + p)$ odd and $n \neq q$, $(n + q)$ odd. These are shown below.

Equation (2.39) for u_o becomes:

$$0 \cdot A_{mn} + 0 \cdot B_{mn} +$$

$$\left\{4mnq\left[2\pi^2 b^2(2p^2 + m^2)F_{16} + 6\pi^2 a^2 n^2 F_{26} +\right.\right.$$

$$\left.\left.3a^2b^2h^2A_{26}\right]/[3\pi ab^2h^2R(p^2 - m^2)(q^2 - n^2)]\right\}C_{mn} +$$

$$\left\{4nq(p^2 + m^2)A_{16}/[(p^2 - m^2)(q^2 - n^2)]\right\}E_{mn} +$$

$$\left\{4mq(b^2p^2A_{16} + a^2n^2A_{26})/[ab(p^2 - m^2)(q^2 - n^2)]\right\}G_{mn} = 0 \tag{2.79}$$

Equation (2.40) for v_o becomes:

$$0 \cdot A_{mn} + 0 \cdot B_{mn} +$$

$$\left\{4mnp\left[2\pi^2 a^2(2q^2 + n^2)F_{26} - 2\pi^2 m^2 b^2 F_{16} -\right.\right.$$

$$\left.\left.3a^2b^2h^2A_{26}\right]/[3\pi a^2bh^2R(p^2 - m^2)(q^2 - n^2)]\right\}C_{mn} +$$

$$\left\{4np(a^2q^2A_{26} + b^2m^2A_{16})/[ab(p^2 - m^2)(q^2 - n^2)]\right\}E_{mn} +$$

$$\left\{4mp(q^2 + n^2)A_{26}/[(p^2 - m^2)(q^2 - n^2)]\right\}G_{mn} = 0 \tag{2.80}$$

Equation (2.41) for w becomes:

$$0 \cdot A_{mn} + 0 \cdot B_{mn} +$$

$$\Big\{mnpq\Big[R^2\Big(256\pi^2(a^2n^2H_{26} + b^2m^2H_{16}) + 72a^2b^2(h^4A_{45} - 8h^2D_{45} +$$

$$16F_{45})\Big) + 32a^2(4\pi^2n^2J_{26} -$$

$$3b^2h^2F_{26})\Big]/[9a^2b^2h^4R^2(p^2 - m^2)(q^2 - n^2)]\Big\}C_{mn} +$$

$$\Big\{4npq\Big[2\pi^2(a^2n^2F_{26} + b^2m^2F_{16}) -$$

$$a^2b^2h^2A_{26}\Big]/[\pi ab^2h^2R(p^2 - m^2)(q^2 - n^2)]\Big\}E_{mn} +$$

$$\Big\{4mpq\Big[\pi^2(6a^2n^2F_{26} - 2b^2m^2F_{16}) -$$

$$3a^2b^2h^2A_{26}\Big]/[3\pi a^2bh^2R(p^2 - m^2)(q^2 - n^2)]\Big\}G_{mn} = 0 \qquad (2.81)$$

Equation (2.42) for ψ_x becomes:

$$\Big\{8mnpq(16H_{16} + 9h^4D_{16} -$$

$$24h^2F_{16})/[9h^4(p^2 - m^2)(q^2 - n^2)]\Big\}A_{mn} +$$

$$\Big\{4mnpq\Big[16(H_{66} + H_{12}) + 9h^4(D_{66} + D_{12}) -$$

$$24h^2(F_{66} + F_{12})\Big]/[9h^4(p^2 - m^2)(q^2 - n^2)]\Big\}B_{mn} +$$

$$0 \cdot C_{mn} + 0 \cdot E_{mn} + 0 \cdot G_{mn} = 0 \qquad (2.82)$$

And finally, Eq (2.43) for ψ_y becomes:

$$\left\{4mnpq\left[16(H_{66} + H_{12}) + 9h^4(D_{66} + D_{12}) -\right.\right.$$

$$\left.\left.24h^2(F_{66} + F_{12})\right]/[9h^4(p^2 - m^2)(q^2 - n^2)]\right\}A_{mn} +$$

$$\left\{8mnpq(16H_{26} + 9h^4D_{26} - 24h^2F_{26})/[9h^4(p^2 - m^2)(q^2 - n^2)]\right\}B_{mn}$$

$$+ 0 \cdot C_{mn} + 0 \cdot E_{mn} + 0 \cdot G_{mn} = 0 \tag{2.83}$$

Equations (2.64) to (2.83) are now ready to be put in matrix format and then input into the eigenvalue subroutine to solve for either ω^2 or $\overline{N}_1$.

III. DISCUSSION AND RESULTS

This chapter will describe the computer program used to calculate the natural frequencies and buckling loads. It will also give physical descriptions of the circular cylindrical shell panels used and will describe the subsequent analysis performed with those panels.

COMPUTER PROGRAM

One FORTRAN program was written for both boundary conditions. Both programs consist of a main program which simply receives the user input data and calls two subroutines that perform the bulk of the work. The first subroutine calculates the stiffness matrix elements in Eq (2.16). The second subroutine uses these stiffness terms to set up the eigenvalue problem, and then calls a subroutine from the IMSL to solve for the eigenvalues and eigenvectors. A complete listing of the program is in Appendix D. The program is discussed in detail below.

The main program, entitled "MAINTHESIS", receives the input data and calls the two subroutines. The following data is collected:

1) An integer flag ("1" or "2"): "1" to perform a vibration problem, and "2" to perform a buckling problem

2) a, length in x direction

3) b, circumferential length in y direction

4) R, radius of curvature

5) h, laminate thickness

6) NPLYS, number of plys in the laminate

7) θ_i, orientation angle of each ply

8) E_1, modulus in the 1 direction

9) E_2, modulus in the 2 direction

10) G_{12}, shear modulus in the 1,2 plane

11) ν_{12}, poisson's ratio

12) ρ, mass density (same for each ply)

13) M = N, maximum number of terms in each admissible function

The main program declares all variables and arrays double precision and allocates workspace for the eigenvalue calculations. The largest problem this program will handle is M=N=10. From Eqs (2.49), (2.50), and (2.62), each admissible function can be approximated by a maximum of 100 terms, resulting in an eigenvalue problem that involves (500x500) matrices. The main program also calculates the following engineering constants:

$$\nu_{21} = \nu_{12} \cdot E_2/E_1$$

$$G_{13} = G_{12}$$

$$G_{23} = 0.8G_{12}$$

It then uses the ply layup information and the engineering constants as input to call the laminated stiffness subroutine.

The laminate stiffness subroutine named "LAMINAT" calculates the extensional, bending, and the higher order stiffness elements of Eq (2.16). The following restrictions apply: the laminate must be symmetric, and only the orientation angle θ_i may change from ply to ply (the density, thickness, and engineering constants remain the same). The subroutine uses the following input data: h, NPLYS, θ_i, E_1, E_2, G_{12}, G_{13}, G_{23}, ρ, ν_{12}, and ν_{21}. It first calculates the reduced stiffness terms, $[Q_{ij}]$, from Eq (2.11). Then it calculates the transformed reduced stiffness terms, $[\bar{Q}_{ij}]$, from Eq (2.13) for each ply by looping from the first ply at the bottom of the laminate to the last ply at the top. For each ply, the extensional, bending, and higher order stiffness terms are calculated and summed together according to Eq (2.16). The output is returned to the main program, printed, and then used as input to the subroutine "GALERK".

Subroutine "GALERK" creates the stiffness and mass/inertia matrices, forms the eigenvalue problem, solves for the eigenvalues and eigenvectors, and determines the mode shape along the midlines of the laminate. In other words, it calculates w(a/2,y) and w(x,b/2). The subroutine is by far the largest portion of the entire program and is also the only part of the whole program that is boundary condition dependant. That is, for the simply supported boundary condition, the subroutine will generate the Galerkin equations using Eqs (2.52) to (2.61); for the clamped boundary condition, the subroutine uses Eqs (2.64)

to (2.83).

The subroutine has four nested DO LOOPS. It cycles through p, q, m, and n and generates the Galerkin equations according to the Cases of integration outlined in Chapter II. At each step in the iteration process, the equations are assembled into matrix format as shown:

$$\begin{array}{l} u_o \rightarrow \\ v_o \rightarrow \\ w \rightarrow \\ \psi_x \rightarrow \\ \psi_y \rightarrow \end{array} \left[\begin{array}{ccc} & & \\ & \text{stiffness matrix} & \\ & & \end{array} \right] \begin{Bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \\ E_{mn} \\ G_{mn} \end{Bmatrix} =$$

$$(\omega^2 \text{ or } \bar{N}_1) \cdot \left[\begin{array}{ccc} & & \\ & \text{mass/inertia matrix} & \\ & & \end{array} \right] \begin{Bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \\ E_{mn} \\ G_{mn} \end{Bmatrix} \tag{3.1}$$

The stiffness matrix is the assemblage of the left hand sides of the Galerkin equations, and the mass/inertia matrix is the assemblage of the right hand sides. Both matrices have $(5 \cdot M \cdot N)$ rows and $(5 \cdot M \cdot N)$ columns. Every value of p and q generates a new row for each of the 5 degrees of freedom, and every value of m and n generates a new column. ω^2 or $\bar{N}_1$ is the eigenvalue; depending upon the integer flag input by the user ("1" for vibration problem, or "2" for buckling problem), the mass/inertia matrix will contain terms associated with either ω^2

or $\bar{N}_1$. The vector:

$$\begin{Bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \\ E_{mn} \\ G_{mn} \end{Bmatrix}$$

is the associated eigenvector. The stiffness and mass/inertia matrices are then input to the IMSL subroutine DGVCRG which calculates the eigenvalues and eigenvectors.

The subroutine then determines the fundamental mode shape along the midlines of the laminate. First it substitutes the C_{mn} coefficients from the eigenvector into the deflection equation, which for both boundary condition considered is:

$$w(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn} \sin(m\pi x/a)\sin(n\pi y/b) \tag{3.2}$$

The circumferential mode shape is determined by calculating values of $w(a/2,y)$, and the longitudinal mode shape is determined from $w(x,b/2)$:

$$\begin{aligned} w(a/2,y) &= \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn} \sin(m\pi/2)\sin(n\pi y/b) \\ w(x,b/2) &= \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn} \sin(m\pi x/a)\sin(n\pi/2) \end{aligned} \tag{3.3}$$

The eigenvalues and mode shape data points are then printed. (Appendix E contains an example for M=N=2 that shows how the Galerkin equations are generated to form the stiffness and mass/inertia matrices.)

ANALYSIS PERFORMED

Several analytical studies were performed to demonstrate the results of this thesis. First, the convergence characteristics of the Galerkin method were demonstrated. Then, a case comparison study with Donnell cylindrical shell panel solutions was performed. The effects of transverse shear deformation, radius of curvature variation, and rotary inertia were investigated. Finally, the influence of varying the length to span ratio was studied.

Laminated Circular Cylindrical Shell Panel Properties

The cylindrical shell panel studied in this thesis is constructed of graphite-epoxy material and has the following material properties:

E_1 = 2.10 E+07 psi

E_2 = 1.40 E+06 psi

G_{12} = 6.00 E+05 psi

ν_{12} = 0.3

ρ = 1.42454 E-04 slugs/in^3 (0.055 lbm/in^3)

Two ply layups were investigated: $[0_{50}/90_{50}]_s$ and $[+45_{50}/-45_{50}]_s$ (both of which for convenience will be referred to as $[0/90]_s$ and $[\pm 45]_s$). The latter ply layup will introduce more shear stiffness terms into the formulation. Tables 3.1 and

3.2 contain the stiffness terms calculated from subroutine "LAMINAT". These stiffness terms will be used in all analysis throughout this chapter, except for the Donnell comparison study.

Table 3.1 Panel Stiffness Elements (h = 1.0 in., $[0/90]_s$)

Extensional Stiffness Elements		
A_{11} = 11267605.634	A_{12} = 422535.211	A_{22} = 11267605.634
A_{16} = 0.0	A_{26} = 0.0	A_{66} = 600000.0
A_{44} = 540000.00	A_{45} = 0.0	A_{55} = 540000.0

Bending Stiffness Elements		
D_{11} = 1555164.319	D_{12} = 35211.268	D_{22} = 322769.953
D_{16} = 0.0	D_{26} = 0.0	D_{66} = 50000.0
D_{44} = 41250.0	D_{45} = 0.0	D_{55} = 48750.0

Higher Order Stiffness Elements		
F_{11} = 256382.042	F_{12} = 5281.690	F_{22} = 25308.099
F_{16} = 0.0	F_{26} = 0.0	F_{66} = 7500.0
F_{44} = 6046.875	F_{45} = 0.0	F_{55} = 7453.125
H_{11} = 46814.088	H_{12} = 943.159	H_{22} = 3487.723
H_{16} = 0.0	H_{26} = 0.0	H_{66} = 1339.286
J_{11} = 9152.886	J_{12} = 183.392	J_{22} = 628.022
J_{16} = 0.0	J_{26} = 0.0	J_{66} = 260.417

Units: A_{ij} are lb/in, D_{ij} are lb·in, F_{ij} are $lb^3 \cdot in$, H_{ij} are $lb^5 \cdot in$, and J_{ij} are $lb^7 \cdot in$

Table 3.2 Panel Stiffness Elements (h = 1.0 in, $[\pm 45]_s$)

Extensional Stiffness Elements		
A_{11} = 6445070.423	A_{12} = 5245070.423	A_{22} = 6445070.423
A_{16} = 0.0	A_{26} = 0.0	A_{66} = 5422535.211
A_{44} = 540000.00	A_{45} = 0.0	A_{55} = 540000.0

Bending Stiffness Elements		
D_{11} = 537089.202	D_{12} = 437089.202	D_{22} = 537089.202
D_{16} = 308098.592	D_{26} = 308098.592	D_{66} = 451877.934
D_{44} = 45000.0	D_{45} = -3750.000	D_{55} = 45000.0

Higher Order Stiffness Elements		
F_{11} = 80563.380	F_{12} = 65563.380	F_{22} = 80563.380
F_{16} = 57768.486	F_{26} = 57768.486	F_{66} = 67781.690
F_{44} = 6750.0	F_{45} = -703.125	F_{55} = 6750.0
H_{11} = 14386.318	H_{12} = 11707.746	H_{22} = 14386.318
H_{16} = 10831.591	H_{26} = 10831.591	H_{66} = 12103.873
J_{11} = 2797.340	J_{12} = 2276.506	J_{22} = 2797.340
J_{16} = 2131.216	J_{26} = 2131.216	J_{66} = 2353.531

Units: A_{ij} are lb/in, D_{ij} are lb·in, F_{ij} are $lb^3\cdot in$,
H_{ij} are $lb^5\cdot in$, and J_{ij} are $lb^7\cdot in$

Galerkin Method Convergence.

As Bowlus (3) points out in his thesis, one needs to determine if the Galerkin technique converges to an exact answer. This section *does not* attempt to prove the convergence of the Galerkin technique. Instead, it shows, as did (24), the necessary (but not sufficient) condition that the frequencies and buckling loads drop by smaller and smaller amounts, approaching exact values asymptotically, as the values of M and N are increased. Referring to Eq (2.49), if M=N=2 then each admissible function is approximated by 4 terms; if M=N=10, there are 100 terms, and so on. M=N=10 was set as the maximum convergence limit due to computer memory limitations and lengthy CPU run times.

Table 3.3 displays the Galerkin convergence characteristics of the natural frequency for several arbitrary panel configurations. Table 3.4 shows the convergence for the critical buckling loads. Finally, Figures 3.1 and 3.2 show plots of the frequency and buckling convergence tendencies for h/R=1/5 and b/h=20.0 for $[\pm 45]_s$ laminates. All data indicates the natural frequencies tend to converge faster than the buckling loads. In other words, vibration problems need lower values of M and N than buckling problems to achieve converged solutions.

The difference in buckling load and natural frequency convergence tendencies is explained by the mode shape of the

deformed panel. In general, for both the frequency and buckling problems, the longitudinal mode shape behaves like that of a flat plate: usually one full sine wave or one half sine wave.

Table 3.3 Galerkin Convergence
Fundamental Frequency (rad/sec)

	b = 10.0 in		b = 15.0 in		b = 20.0 in	
M = N	ω	Decrease	ω	Decrease	ω	Decrease
2	30840.4	----	20320.5	----	15647.9	----
4	27738.5	10.1%	16546.3	18.6%	11866.7	24.2%
6	27608.4	0.47%	16383.5	0.99%	11711.7	1.31%
8	27598.1	0.04%	16342.7	0.25%	11665.7	0.39%

Clamped Boundary Condition
$[\pm 45]_s$, R = 5.0 in, h = 1.0 in, a/b=1

w(x,b/2) in Eq (3.3) takes on this shape as x varies from 0 to a.

The circumferential mode shape, w(a/2,y), behaves differently for the two problems. For vibration, there is generally 1.0 to 1.5 full sine waves in the circumferential direction from y = 0 to y = b. Interestingly, the mode shape is independent of the degree of accuracy chosen: M=N=2 generally produces the same shape as M=N=10.

On the other hand, the buckling mode shape depends a great deal upon the degree of accuracy. The general trends can be explained by an example. Refer to table 3.4, clamped boundary

condition, R = 5 in, b = 20 in. M=N=2 generates $\bar{N}_1$=1388936.8 lb/in. The circumferential mode shape is a single half sine wave. For M=N=10, $\bar{N}_1$=309620.0 lb/in (4.5 times lower), and there are six full sine waves in the circumferential direction.

Table 3.4 Galerkin Convergence
Critical Bucklin Load (lb/in)

	b = 10.0 in		b = 30.0 in	
M = N	$\bar{N}_1$	Decrease	$\bar{N}_1$	Decrease
2	542522.3	------	800558.9	------
4	380559.9	29.9%	603021.4	24.7%
6	360626.9	5.2%	382408.0	36.6%
8	359194.1	0.4%	329291.8	13.9%
10	358660.4	0.1%	305460.9	7.2%

Simply supported boundary condition
$[\pm 45]_s$, h=1.0 in, a/b=1, R=5.0 in

	b = 20.0 in		b = 30.0 in	
M = N	$\bar{N}_1$	Decrease	$\bar{N}_1$	Decrease
2	1388936.8	------	1720640.7	------
4	488639.3	64.8%	764410.9	55.6%
6	347096.8	29.0%	449037.7	41.3%
8	311436.4	10.3%	340343.2	24.2%
10	309620.0	0.6%	304491.6	10.5%

Clamped boundary condition
$[\pm 45]_s$, h=1.0 in, a/b = 1, R=5.0 in

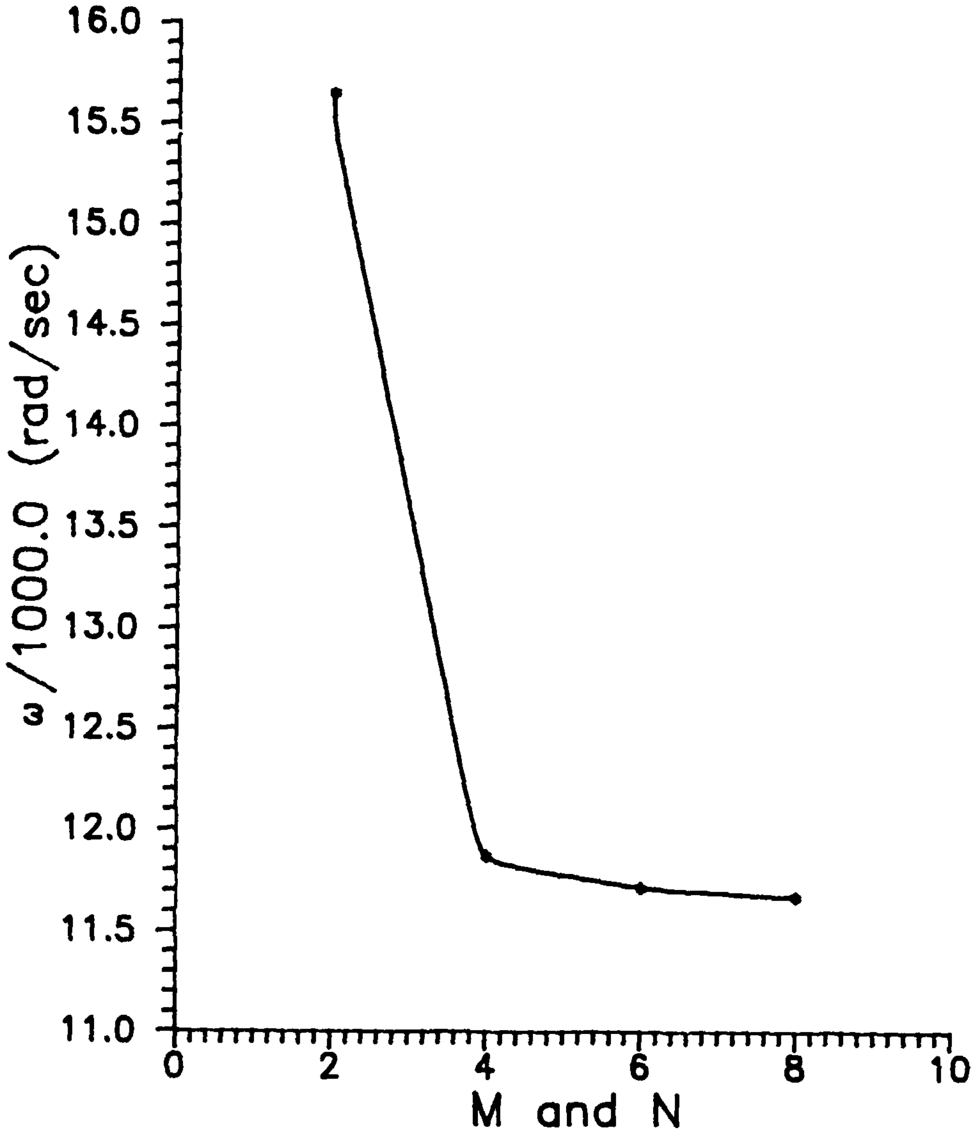

Figure 3.1 Fundamental Frequency Convergence

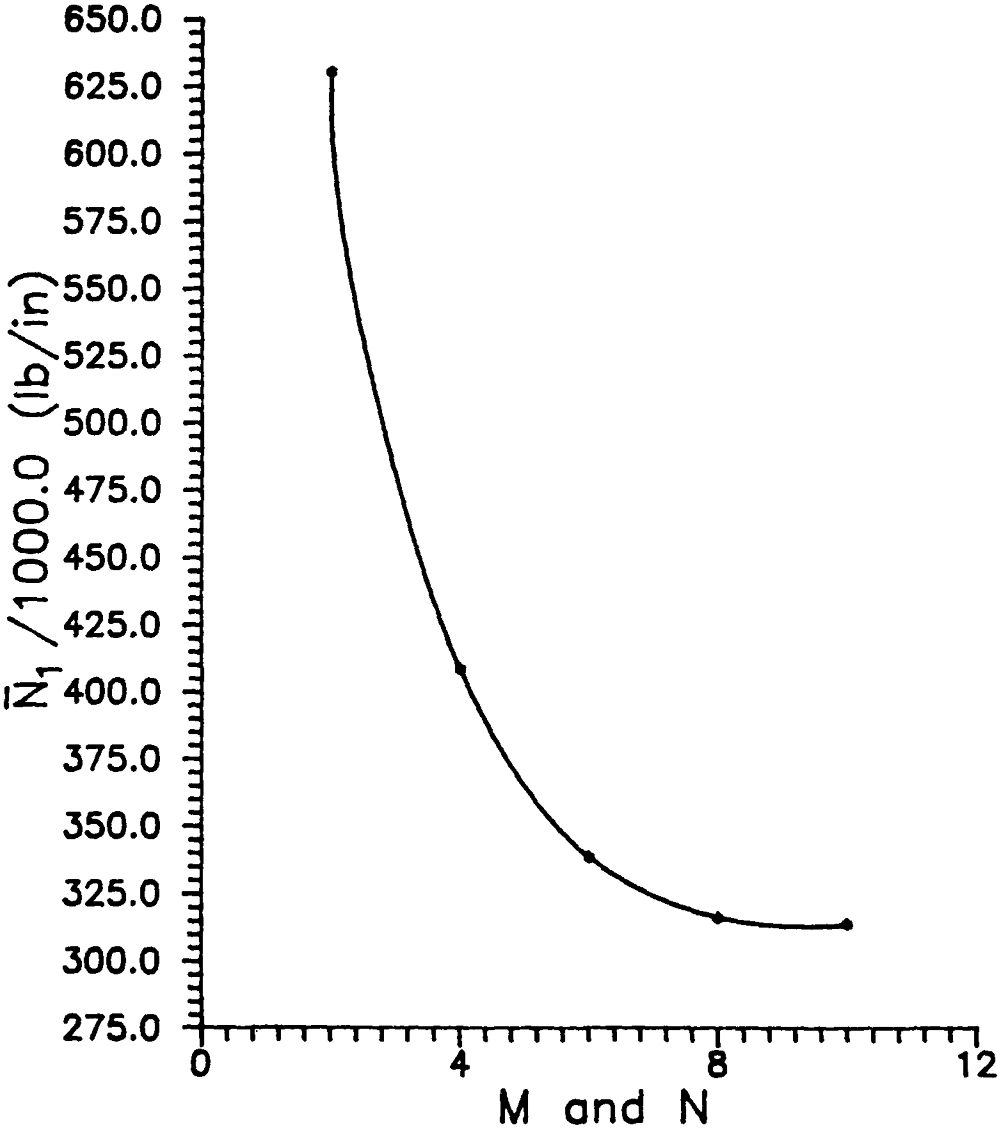

Figure 3.2 Critical Buckling Load Convergence

Bushnell (6) found this same type of mode shape behavior for axially compressed cylindrical shells. It is clear that more terms are needed to accurately model the buckling mode shape, which explains the slower convergence tendencies for certain geometries.

All panel configurations used in this thesis displayed excellent frequency and buckling load convergence towards exact answers. This data does not prove convergence, but it definitely demonstrates convergence tendencies. The drawback with the Galerkin technique is that in order to obtain extremely accurate answers that require M and N be greater than 10, a great deal of computer resources is required. This higher accuracy requirement has more application with the buckling loads, since they don't converge as fast as the natural frequencies.

Comparison Study With Donnell Solution.

Reddy and Liu (16) examined laminated circular cylindrical shell panels using Donnell theory with parabolic transverse shear modeling. The equations of motion for the circular cylindrical shell panel, Eq (2.33), will degenerate down to the Donnell equations of motion by dropping the appropriate higher order terms in Eq (2.33a) as previously discussed in Chapter II. Reddy found an exact solution to the equations of motion for

simply supported boundary conditions using Navier's method. The Navier solution exists only if the following stiffnesses are equal to zero: $A_{i6} = D_{i6} = F_{i6} = H_{i6} = 0$ $(i = 1,2)$ and $A_{45} = D_{45} = F_{45} = 0$. This restricts his analysis to panels with $[0/90]_s$ ply layups.

Reddy used different engineering constants in his work than those used in this thesis. The following values were used in the comparison:

$E_1 = 2.1$ E+07 psi

$E_2 = 8.4$ E+05 psi

$G_{12} = 4.2$ E+05 psi

$G_{13} = G_{12}$

$G_{23} = 1.68$ E+05 psi

$\nu_{12} = 0.25$

$\rho = 1.0$ slugs/in^3 (Note an extremely large value)

These numbers give the 1.0 in. thick cylindrical shell panel the stiffness terms shown in Table 3.5. Table 3.6 compares Reddy's answers using the Navier solution with those of this thesis using the Galerkin technique. Note that Donnell theory limits the maximum h/R ratio to be about 1/50, as discussed in Chapter II.

Table 3.5 Stiffness Elements for the Reddy Comparison
(h = 1.0 in, $[0/90]_s$)

Extensional Stiffness Elements		
A_{11} = 10947368.421	A_{12} = 210526.316	A_{22} = 10947368.421
A_{16} = 0.0	A_{26} = 0.0	A_{66} = 420000.0
A_{44} = 294000.00	A_{45} = 0.0	A_{55} = 294000.0

Bending Stiffness Elements		
D_{11} = 1543859.649	D_{12} = 17543.860	D_{22} = 280701.754
D_{16} = 0.0	D_{26} = 0.0	D_{66} = 35000.0
D_{44} = 16625.0	D_{45} = 0.0	D_{55} = 32375.0

Higher Order Stiffness Elements		
F_{11} = 255263.158	F_{12} = 2631.579	F_{22} = 18421.053
F_{16} = 0.0	F_{26} = 0.0	F_{66} = 5250.0
F_{44} = 2198.438	F_{45} = 0.0	F_{55} = 5151.563
H_{11} = 46640.038	H_{12} = 469.925	H_{22} = 2232.143
H_{16} = 0.0	H_{26} = 0.0	H_{66} = 937.50
J_{11} = 9120.294	J_{12} = 91.374	J_{22} = 382.630
J_{16} = 0.0	J_{26} = 0.0	J_{66} = 182.292

Units: A_{ij} are lb/in, D_{ij} are lb·in, F_{ij} are $lb^3 \cdot in$,
H_{ij} are $lb^5 \cdot in$, and J_{ij} are $lb^7 \cdot in$

Table 3.6 Donnell Frequency Comparison

Fundamental Frequency (rad/sec)

a = b = 100.0 in.				
R (in)	$\frac{h}{R}$	Navier	Galerkin	Error (%)
500.0	.002	1.86602	1.8697	+ .2
1000.0	.001	1.52416	1.52458	+ .03
2000.0	.0005	1.42519	1.42529	+ .007
5000.0	.0002	1.39585	1.39623	+ .03
a = b = 10.0 in.				
R (in)	$\frac{h}{R}$	Navier	Galerkin	Error (%)
50.0	.02	108.4237	108.6415	+ .2
100.0	.01	108.0571	108.109	+ .05
200.0	.005	107.9655	107.9753	+ .009
500.0	.002	107.9655	107.9379	- .03

Simply Supported Boundary Condition
$[0/90]_s$, h = 1.0 in.

Table 3.6 validates the accuracy of the higher order theory as it applies to Donnell type problems. The excellent agreement between the higher order theory and the Donnell equations is attributed to the h/R region involved. As explained earlier, since the maximum h/R value is 1/50, the higher order terms in Eq (2.33a) approach zero; the higher order equations of motion reduce to Reddy's Donnell equations. Table 3.6 also shows the Galerkin technique to be an excellent approximation method.

Transverse Shear Deformation Analysis.

This section will compare the parabolic transverse shear model used in this thesis with the Mindlin transverse shear model and the classical Kirchhoff model, which doesn't incorporate transverse shear. This section is devoted to flat plate comparisons, $R \to \infty$.

Classical plate theory with no modeling of transverse shear makes the plate too stiff; consequently, the theory overpredicts the natural frequencies and buckling loads. Therefore, plates modeled with transverse shear will have lower frequencies and buckling loads than Classical plates. Furthermore, the more accurate parabolic transverse shear model should have lower frequencies and buckling loads than the Mindlin model. These statements are verified by comparing the parabolic solutions with Bowlus' (3),(4) and Palardy's (13) Mindlin solutions and Jones' (9) Classical solutions. Bowlus used the Galerkin method, and Palardy used the Levy method to arrive at their flat plate answers. Jones has closed form solutions for the buckling loads and natural frequencies of simply supported specially orthotropic laminated plates using Classical thin plate theory, ie, the Kirchhoff assumptions. The fundamental frequency for a classical thin plate with a ply layup of $[0/90]_s$ is:

$$\omega = \frac{\pi^2}{\rho^{0.5}}\left[D_{11}/a^4 + 2(D_{12} + 2D_{66})/(a^2b^2) + D_{22}/b^4\right]^{0.5} \qquad (3.4)$$

If the aspect ratio, a/b, is greater than 2.5, the critical buckling load for this laminate is:

$$\bar{N}_1 = \frac{2\pi^2}{b^2}\left[D_{12} + 2D_{66} + (D_{11}D_{22})^{0.5}\right] \qquad (3.5)$$

The bending stiffness terms in these two equations are obtained from Table 3.1.

Table 3.7 compares the natural frequencies of the parabolic and Mindlin transverse shear models with the Classical model using Eq (3.4). Table 3.8 compares the parabolic buckling loads with the classical buckling loads using Eq (3.5). Both tables display the expected results, but with one exception;

Table 3.7 Classical Frequency Comparison

Fundamental Frequency (rad/sec)

a (in)	Parabolic	Mindlin	Classical	Classical Error
5.0	32459.7	33210.67	48481.45	+ 49.3 %
10.0	10519.85	10657.15	12120.36	+ 15.2 %
20.0	2910.64	2913.32	3030.09	+ 4.1 %
30.0	1322.20	1322.56	1346.71	+ 1.9 %
40.0	749.66	749.47	757.52	+ 1.0 %
50.0	481.57	483.29	484.81	+ 0.7 %

Simply Supported Boundary Condition

$R = \infty$, $[0/90]_s$, h = 1.0 in, a/b = 1

Table 3.8 Classical Buckling Comparison

Critical Buckling Load (lb/in)

b (in)	Parabolic	Classical	Error
5.0	432811.14	666161.06	+ 53.9 %
10.0	146354.07	166540.27	+ 13.8 %
20.0	40243.73	41635.07	+ 3.5 %
30.0	18226.55	18504.47	+ 1.5 %
40.0	10321.34	10408.77	+ 0.8 %
50.0	6626.29	6661.61	+ 0.5 %

Simply Supported Boundary Condition

$R \rightarrow \infty$, $[0/90]_s$, h = 1.0 in, a/b = 3

Ironically, there is almost a perfect match between the frequencies for the parabolic and Mindlin cases. This is probably because the simply supported boundary condition is so "ideal". There are very good convergence characteristics; in fact, for the $[0/90]_s$ ply layup, the convergence is immediate for M = N = 2. For both the frequency and buckling cases, the transverse shear models approach the classical solutions asymptotically at a/h (or b/h) values of 40.0 to 50.0. See Figures 3.3 and 3.4.

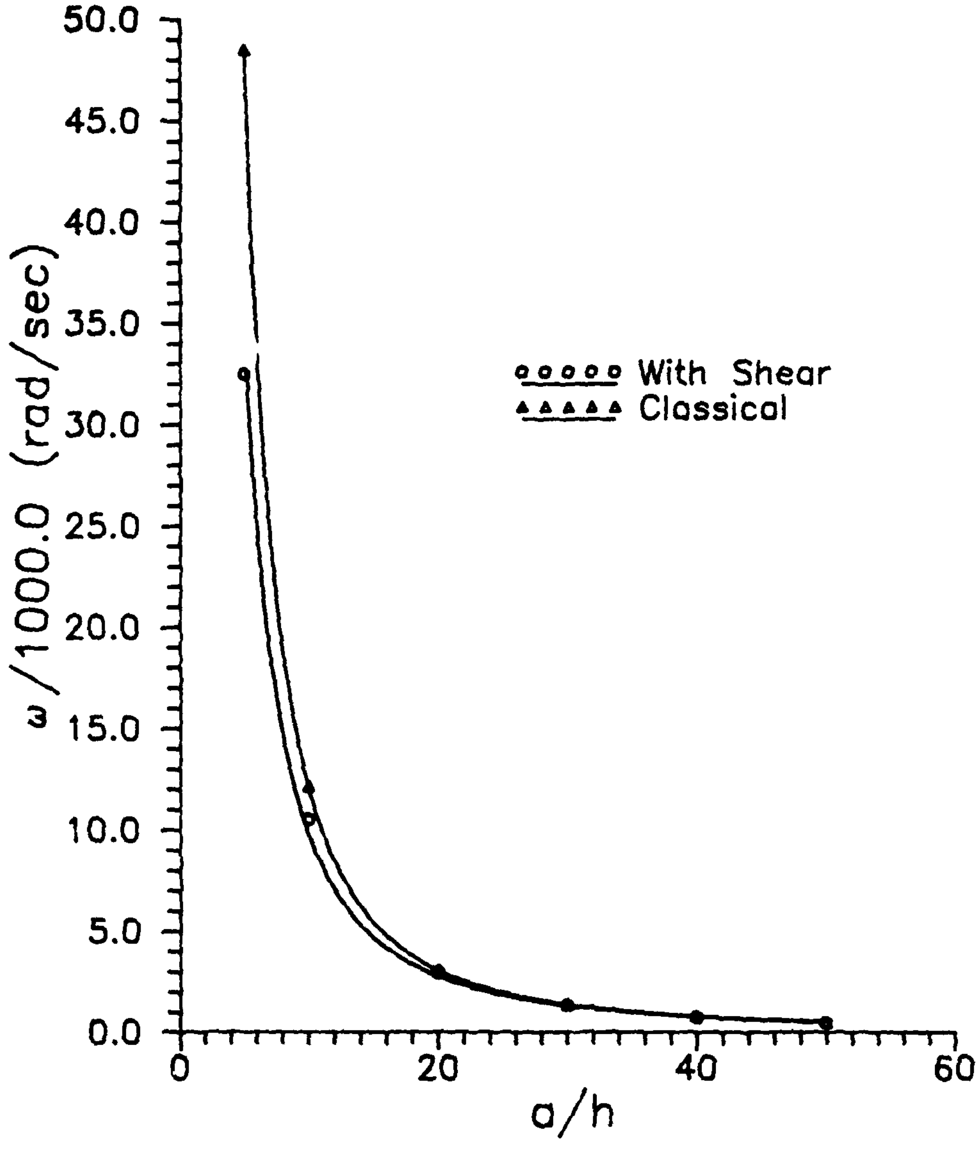

Figure 3.3 Transverse Shear vs Classical Frequencies

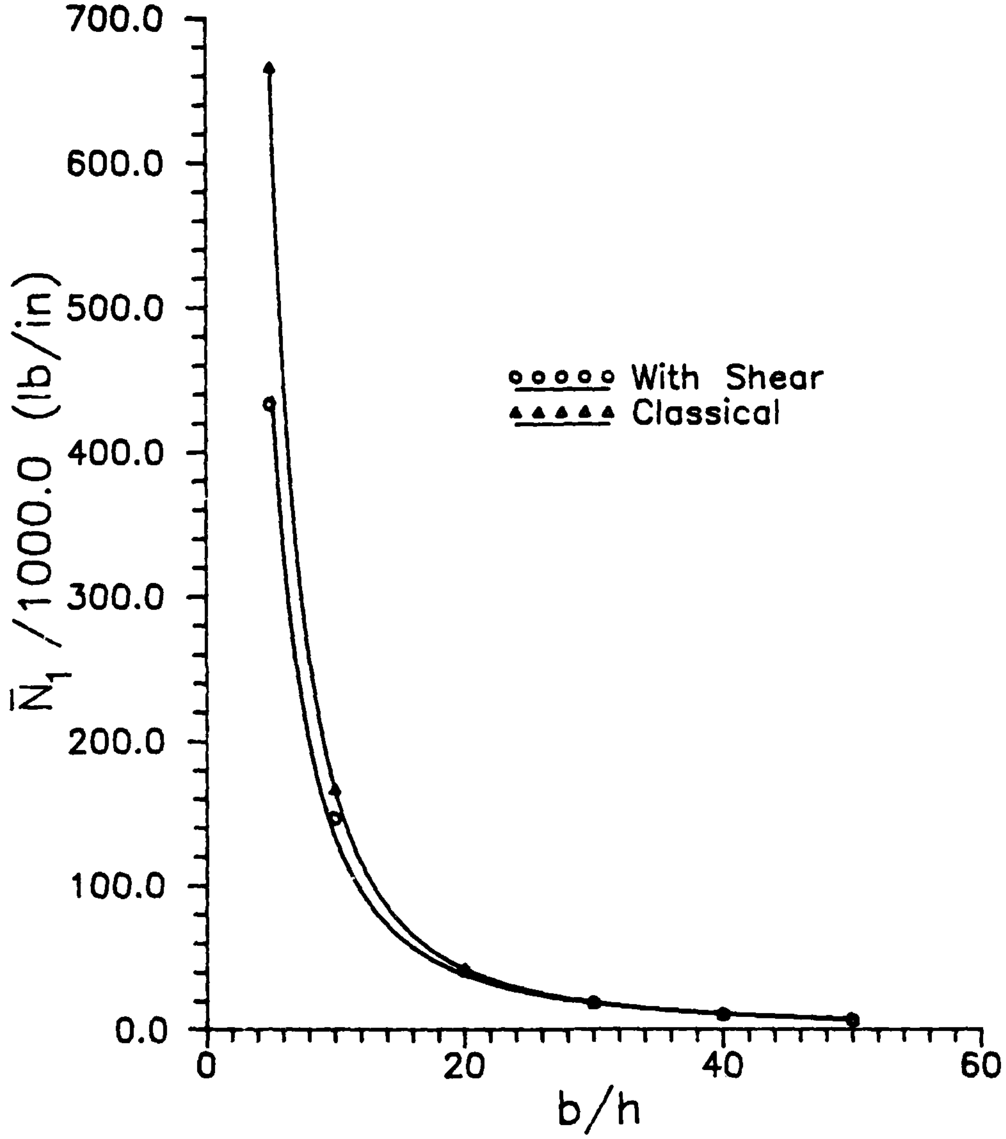

Figure 3.4 Transverse Shear vs Classical Buckling Loads

Tables 3.9 and 3.10 compare parabolic transverse shear frequency solutions with those of Mindlin theory for $[\pm 45]_s$ laminates. Table 3.9, the simply supported case, shows the same trend as before; parabolic and Mindlin solutions are virtually equal.

Table 3.9 Parabolic vs Mindlin Shear Models
Simply Supported Boundary Condition

Fundamental Frequency (rad/sec)

M = N	Parabolic	Mindlin	Error (%)
2	3656.39	3643.01	- .37
4	3596.80	3573.37	- .65
6	3571.40	3539.91	- .88
8	3557.18	3520.58	- 1.03

$R \to \infty$, $[\pm 45]_s$, h = 1.0 in, a=b=20.0 in

The expected departure of the Mindlin theory from the parabolic theory is evident for the clamped boundary condition in Table 3.10. For each value of M and N, the parabolic model produces more accurate frequencies.

Table 3.10 Parabolic vs Mindlin Shear Models
Clamped Boundary Condition

Fundamental Frequency (rad/sec)

$[0/90]_s$			
M = N	Parabolic	Mindlin	Error (%)
2	6609.30	7869.20	+ 19.0
6	5349.30	5698.35	+ 6.5
8	5333.73	5644.82	+ 5.8
$[\pm 45]_s$			
M = N	Parabolic	Mindlin	Error (%)
2	6537.40	7705.62	+ 17.9
6	5098.60	5542.46	+ 8.7
8	5058.90	5231.92	+ 3.4

$R \to \infty$, h = 1.0 in, a = b = 20.0

Radius of Curvature Analysis.

In this section the effects of varying the radius of curvature, R, (or, equivalently, h/R) is examined. As noted in Chapter II, for a flat plate h/R = 0, and membrane completely decouples from bending. The membrane Galerkin equations, those

associated with u_o and v_o, are coupled together, but as a whole are decoupled from the bending Galerkin equations: those associated with w, ψ_x, and ψ_y. As h/R is increased from 0 up to the maximum value of 1/5, membrane and bending couple together, the cylindrical panel becomes deeper and stiffer, and the natural frequencies and buckling loads increase. The following specific h/R ratios are investigated:

$$\frac{h}{R} = \begin{cases} 0: & \text{Flat Plate} \\ 1/50: & \text{Donnell Equation Maximum Limit} \\ 1/20: & \text{Intermediate Value} \\ 1/5: & \text{Maximum Limit of Higher Order Theory} \end{cases}$$

The figures in this section are plots of ω or $\bar{N}_1$ vs. b/h. The panels are square, a/b = 1, and M = N = 6 is used as the degree of accuracy. (The two buckling load plots for h/R = 1/5 required M = N = 8 for b/h values of 5, 10, and 15 and M=N=10 for b/h values of 20 and 30 to obtain proper convergence.) The circumferential length to thickness ratio (b/h) is varied from 5.0 to 50.0.

The four fundamental frequency plots are shown in Figures 3.5 through 3.8. All curves follow the same basic trends: the frequencies are high at b/h = 5.0 and then decrease as the panel gets thinner, approaching constant values asymptotically at b/h values of 40 to 50. Also, as expected, the frequencies increase due to membrane and bending coupling as h/R is increased from 0 to 1/5. The effect of this coupling is shown in Table 3.11 in which panel to flat plate frequency comparisons are made.

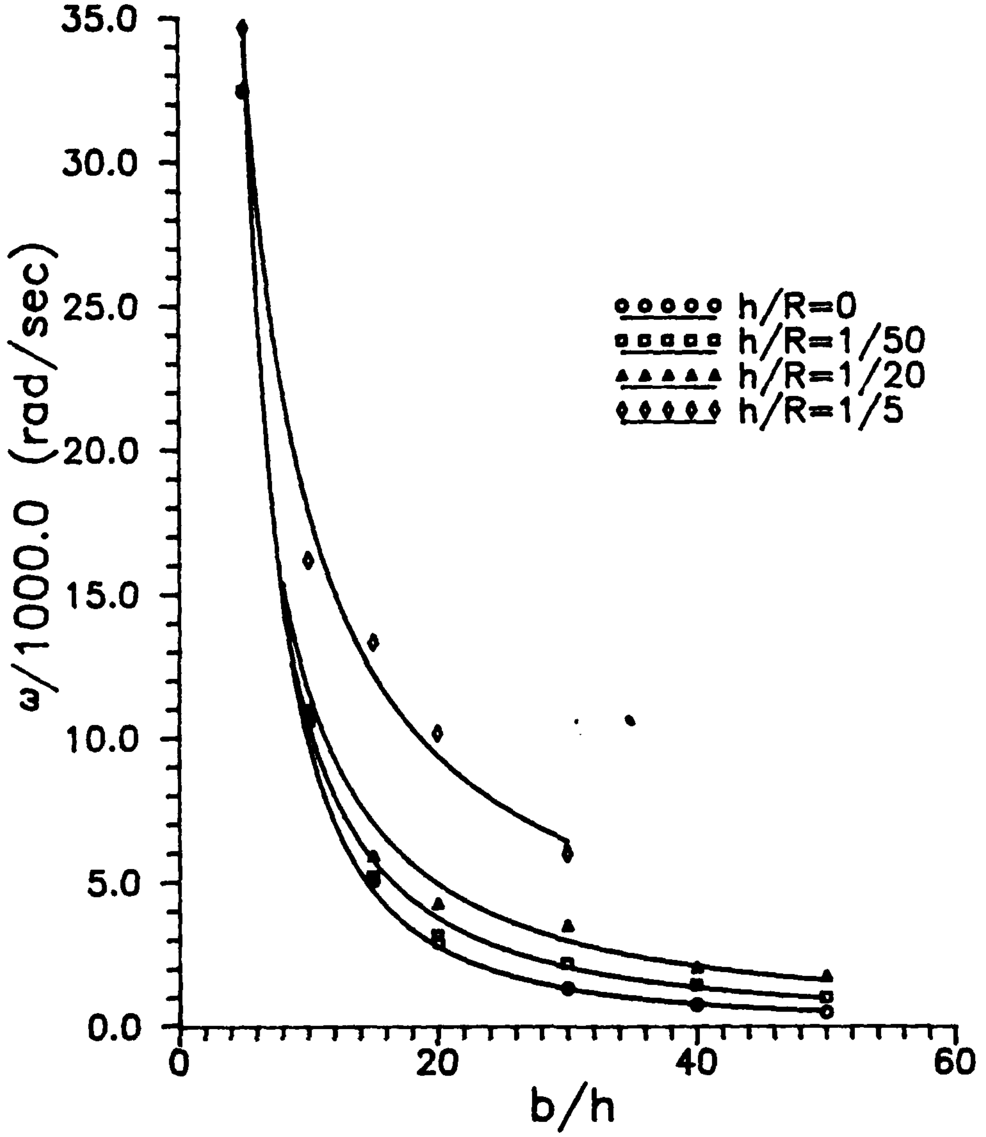

Figure 3.5 Curvature Effects on Frequency Simply Supported Boundary, $[0/90]_s$

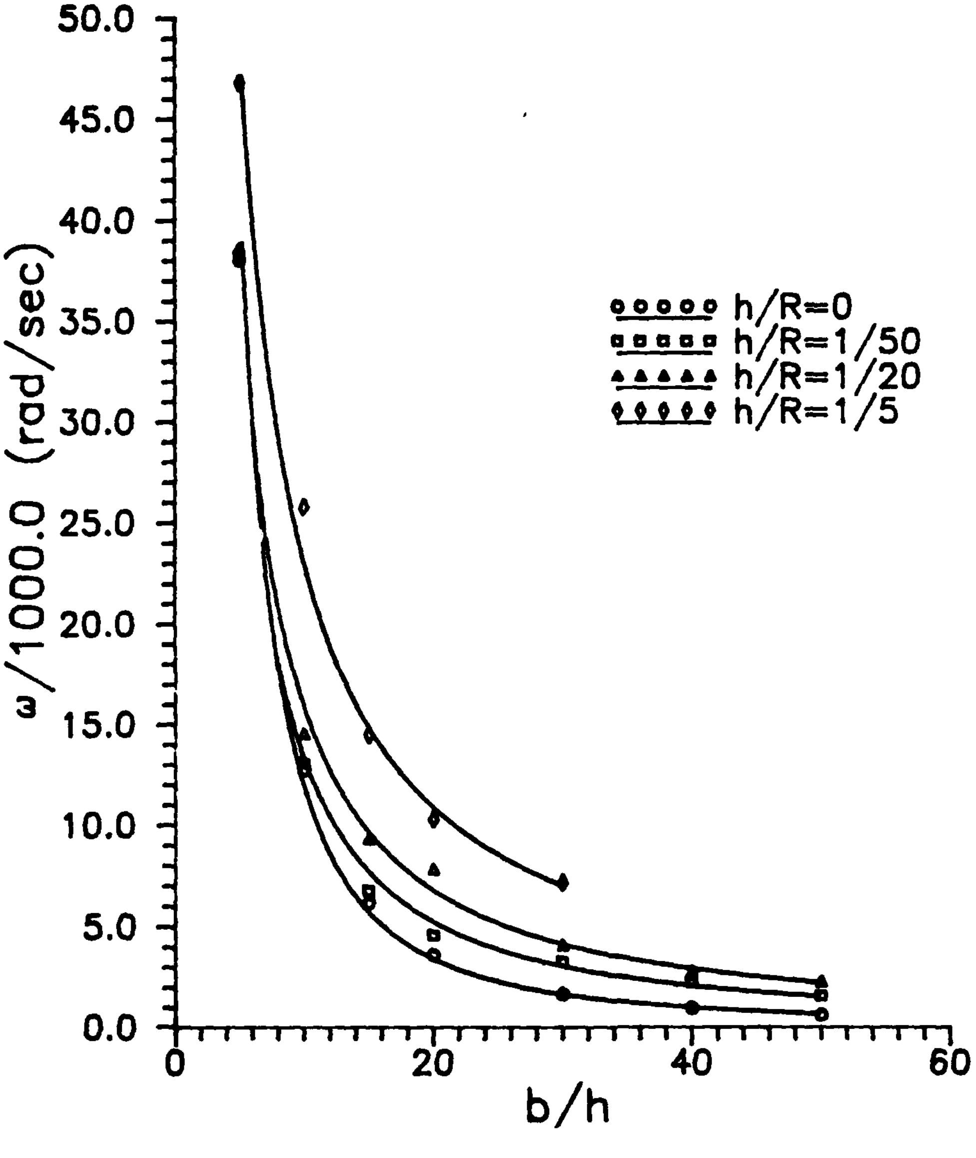

Figure 3.6 Curvature Effects on Frequency Simply Supported Boundary, $[\pm 45]_S$

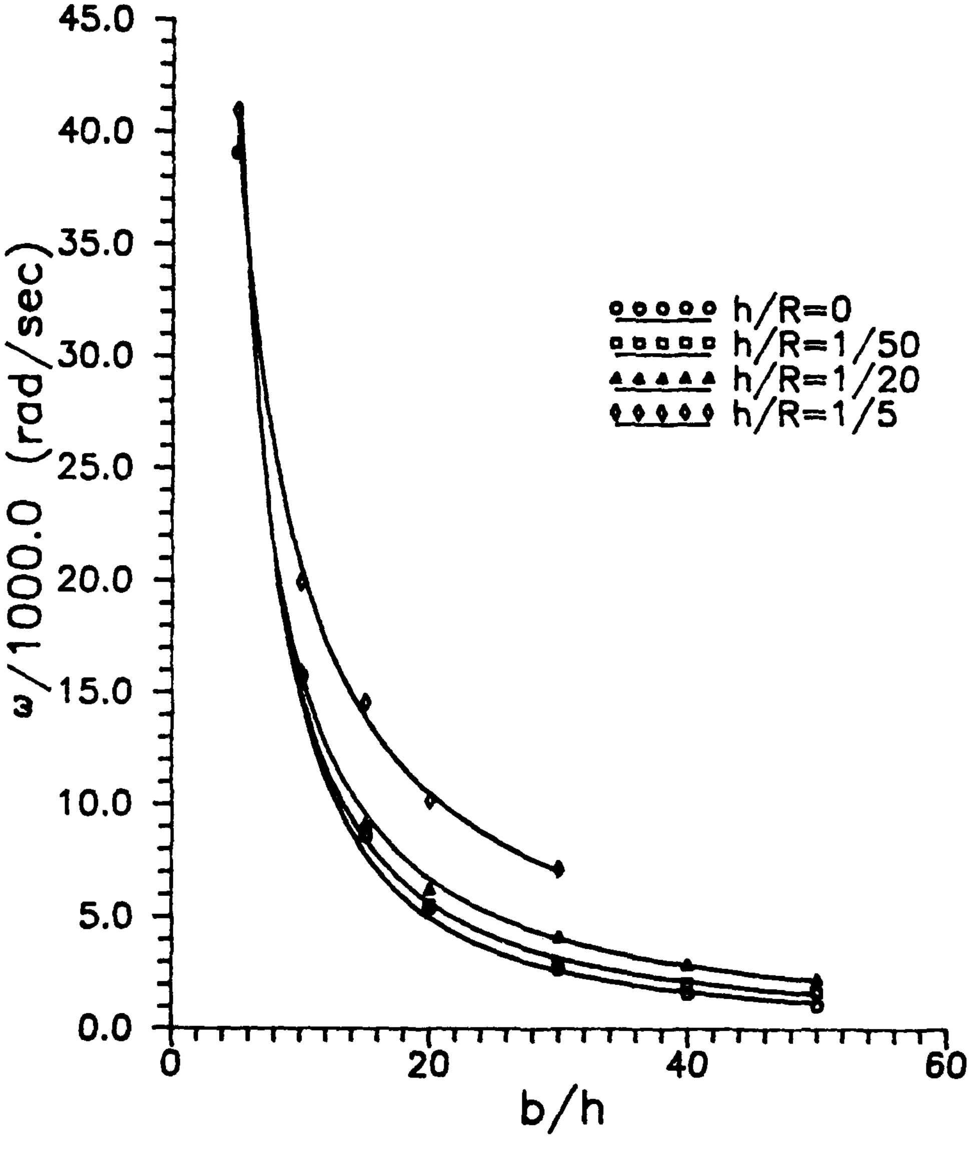

Figure 3.7 Curvature Effects on Frequency Clamped Boundary, $[0/90]_s$

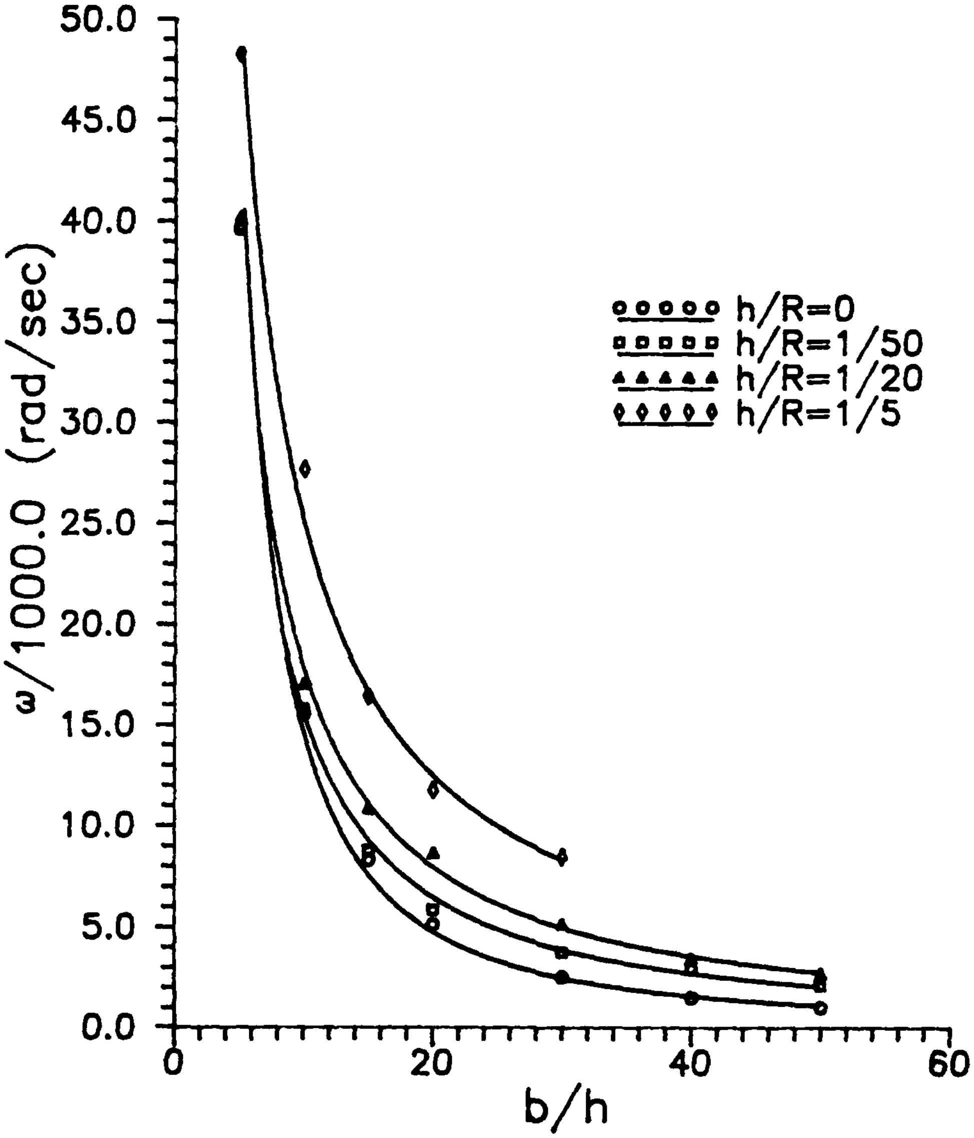

Figure 3.8 Curvature Effects on Frequency
Clamped Boundary, $[\pm 45]_s$

Table 3.11 Frequency Coupling Effects

ω/ω_p

	$[0/90]_s$		$[\pm45]_s$	
h/R	b/h = 10	b/h = 30	b/h = 10	b/h = 30
1/5	1.54	4.52	2.02	4.37
1/20	1.04	2.65	1.14	2.48

Simply Supported Boundary Conditions

	$[0/90]_s$		$[\pm45]_s$	
h/R	b/h = 10	b/h = 30	b/h = 10	b/h = 30
1/5	1.27	2.67	1.78	3.37
1/20	1.02	1.54	1.10	2.05

Clamped Boundary Conditions

ω_p = flat plate natural frequency

h = 1.0 in, a/b = 1

With few exceptions, the $[\pm45]_s$ laminates generally yield higher frequencies than the $[0/90]_s$ laminates for both boundary conditions considered. The $[\pm45]_s$ laminates have inplane shear stiffness terms (D_{16}, D_{26}, F_{16}, F_{26}, H_{16}, H_{26}, J_{16}, J_{26}) that account for these higher frequencies. (See Table 3.1 and 3.2.) Referring to Figures 3.5 and 3.6, for the simply supported boundary condition, the difference in frequency for the two laminates gets greater as the curvature increases. For h/R = 0, the frequencies are about 20% higher for the $[\pm45]_s$ laminate for

all b/h values. For $h/R = 1/50$, the frequencies are about 20% higher at $b/h = 5$ and are about 50% higher at $b/h = 50$. For $h/R = 1/20$, the frequencies vary from 20% higher to 80% higher, and for $h/R = 1/5$ the frequencies are about 25% higher for all b/h values.

For the clamped boundary condition, the $[0/90]_s$ laminate yields higher frequencies than the $[\pm 45]_s$ laminate for flat plates ($h/R=0$). But, as the curvature increases, the frequencies of the $[\pm 45]_s$ laminate become greater than those of the $[0/90]_s$ laminate. (See Figures 3.7 and 3.8.) For $h/R=1/50$, the frequencies for both laminates are roughly equal at $b/h=5$, but the frequencies are about 30% higher for the $[\pm 45]_s$ laminate at $b/h=50$. For $h/R = 1/20$ the frequencies vary from roughly 6% higher to 20% higher, and for $h/R = 1/5$ the frequencies are roughly 30% higher.

Table 3.11 displays a consistent trend mentioned in the previous two paragraphs; in general, as the curvature of the panel increases, the membrane and bending coupling has a greater effect at larger b/h values. Larger b/h values physically equate to greater arclength around the panel. In fact, for $h/R = 1/5$ at $b/h = 30$ the cylindrical panel is almost a complete cylinder.

For the same ply layup, the frequencies for the clamped boundary condition are higher than those for the simply supported boundary condition. For the $[0/90]_s$ laminate the difference is quite dramatic. For h/R values of 0 and 1/50, the

frequencies are 20% higher at b/h = 5 and over 100% higher at b/h =40 to 50. For h/R = 1/20 the frequencies are roughly 30% higher for all b/h values, and for h/R = 1/5 the frequencies are roughly 20% higher. Figures 3.6 and 3.8 show the same trend for the $[\pm 45]_s$ laminate, but the increases in frequency for the clamped boundary condition over the simply supported boundary condition are not quite as large as they were for the $[0/90]_s$.

Figures 3.9 and 3.10 show the two buckling plots done for the $[\pm 45]_s$ laminate. The same trends of the frequency plots are evident; high buckling loads at small b/h values, decreasing asymptotically to constant loads at b/h values of 40 to 50. There are very significant increases in the buckling loads as h/R is increased, and as Table 3.12 shows, the membrane and bending coupling has a greater *effect* as the circumferential distance around the panel increases.

The $[\pm 45]_s$ laminate yields higher buckling loads for the clamped boundary condition than for the simply supported boundary condition for h/R values of 0, 1/50, and 1/20. The buckling loads are roughly the same for both boundary conditions for h/R = 1/5. Similarly, as Table 3.13 indicates, the $[0/90]_s$ laminate yields buckling loads for the clamped boundary condition that are upwards of 50% higher than the buckling loads for the simply supported boundary condition.

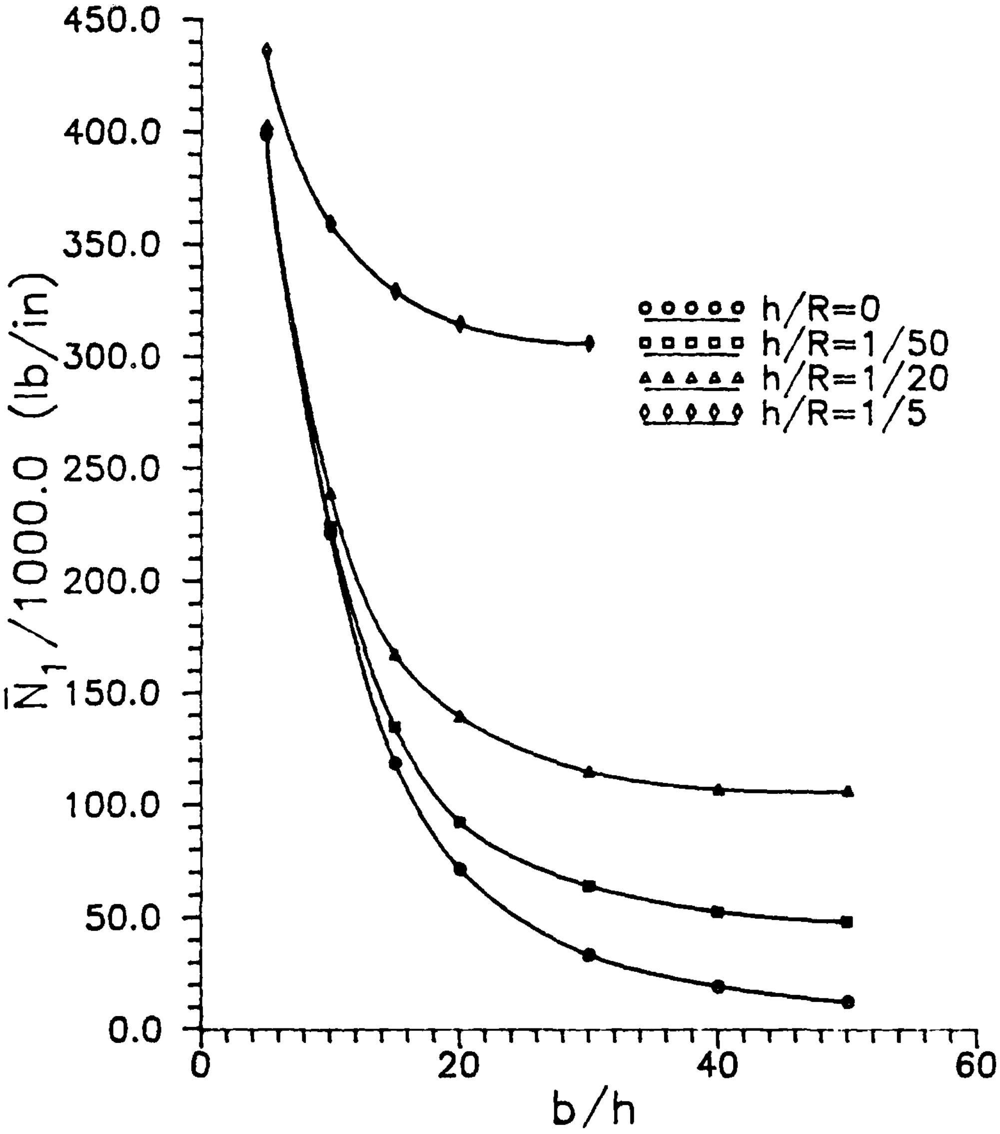

Figure 3.9 Curvature Effects on Buckling Load Simply Supported Boundary, $[\pm 45]_s$

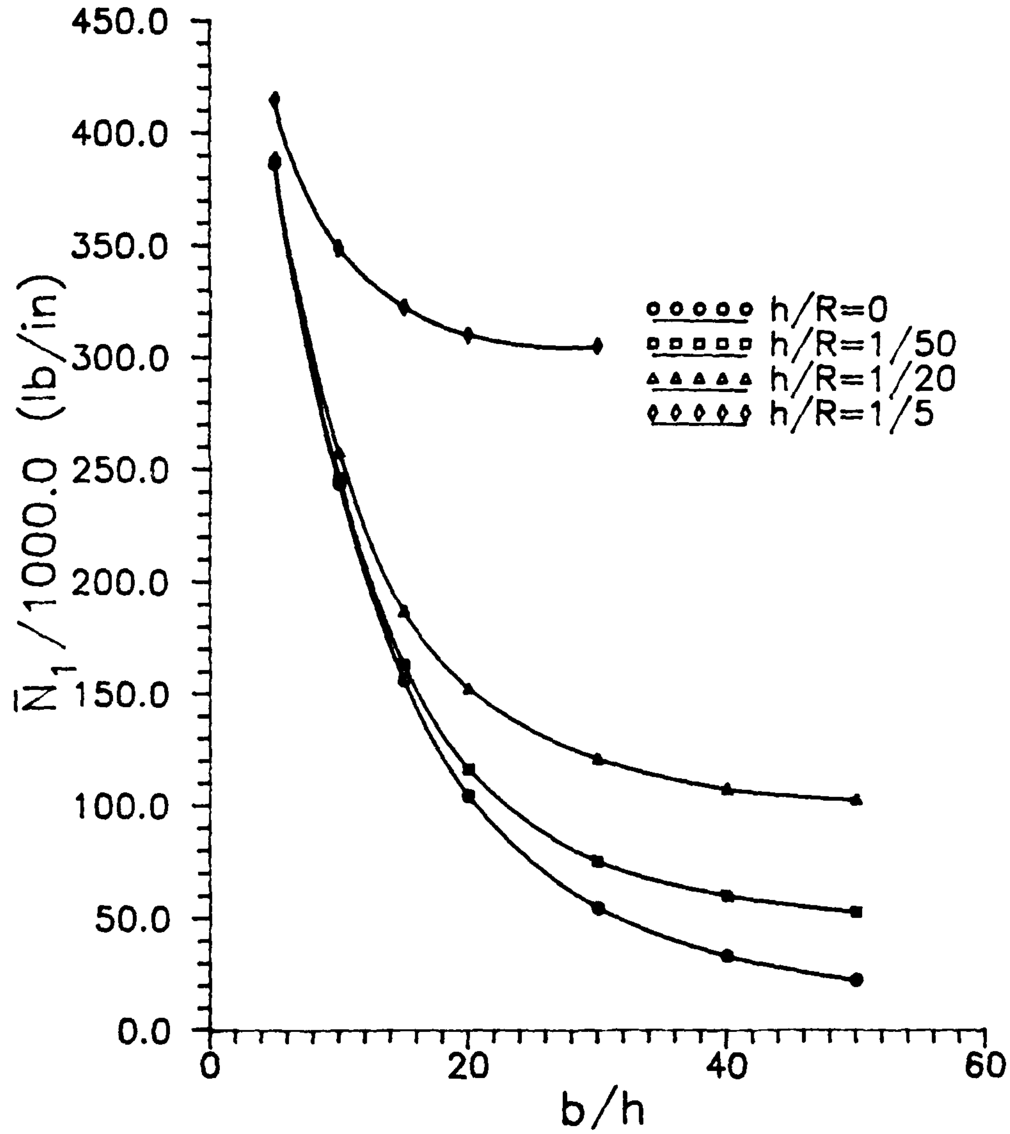

Figure 3.10 Curvature Effects on Buckling Load Clamped Boundary, $[\pm 45]_s$

Table 3.12 Buckling Load Coupling Effects

$\bar{N}_1/\bar{N}_{1_p}$

	Simply Supported		Clamped	
h/R	b/h = 10	b/h = 30	b/h = 10	b/h = 30
1/5	1.63	9.35	1.43	5.61
1/20	1.08	3.45	1.06	2.22

$\bar{N}_{1_p}$ = flat plate buckling load

h = 1.0 in, a/b = 1, $[\pm 45]_s$

Table 3.13 Buckling Loads for $[0/90]_s$ Laminates

Critical Buckling Load (lb/in)

b/h	Simply Supported	Clamped
5	392281.3	454819.2
20	104118.3	195617.0
50	94020.0	134723.6

R = 20.0 in, h = 1.0 in, a/b = 1

Rotary Inertia Analysis.

Another feature of this thesis is the incorporation of rotary inertia into the vibration problem. Referring to Eqs (2.32) and (2.33), the following accelerations contribute to the rotary inertia: $\ddot{w}_{,x}$, $\ddot{w}_{,xx}$, $\ddot{w}_{,y}$, $\ddot{w}_{,yy}$, $\ddot{\psi}_x$, $\ddot{\psi}_{x,x}$, $\ddot{\psi}_y$, $\ddot{\psi}_{y,y}$. If rotary inertia is eliminated, the only inertia term left is $I_1\ddot{w}$ on the right hand side of the equation of motion for w in Eq (2.33). Likewise, the Galerkin equations all reduce to a single term on the right hand sides. The end result is a much less populated mass/inertia matrix in Eq (3.1).

Bowlus (4) and Palardy (13) both found Rotary inertia to be negligible for the vibration of flat plates modeled with Mindlin transverse shear theory. The results are the same for this thesis. Several cases were run for both simply supported and clamped boundary conditions, which included various h/R ratios, ranging from 0 (flat plate) to 1/5, and various a/h (b/h) ratios. The results were all the same. With rotary inertia removed, the fundamental frequencies are only about 0.5% higher. The overall conclusion is rotary inertia has a negligible effect for first mode analysis. It does become more important for the higher modes, however.

Transverse Normal Stress Considerations.

The transverse normal stress, σ_z, was set equal to zero under the assumptions of plain stress constitutive relations. As explained in Chapter II, this is a good assumption for most geometries, and is therefore used quite extensively in plate/panel analyses. Some of the geometries analyzed in the previous sections "stretch" the accuracy limitations of the σ_z assumption and warrant specific comments. First, the flat plate is examined.

For flat plates, the validity of assuming $\sigma_z = 0$ is dependant upon the minimum value of a/h (or b/h) chosen. Koiter (10) states that the transverse normal stress is in general of order h^2/L^2 times the bending stress, and transverse shear strain is h/L times the bending stress, where L is the smallest wavelength of the deformation pattern on the mid surface. For plates, L is almost always equal to the smallest dimension (a or b). Therefore, for the plate:

$$\begin{aligned} \sigma_z &\approx h^2/a^2\,(\sigma_x,\sigma_y) \\ \tau_{xz},\tau_{yz} &\approx h/a\,(\sigma_x,\sigma_y) \end{aligned} \tag{3.6}$$

A rule of thumb in classical plate theories is the minimum a/h ratio is 10, or $\sigma_z \approx 0.01(\sigma_x,\sigma_y)$. References (15), (17), (3), and this author used lower values. Referring to the previous

sections, the minimum a/h (b/h) ratio used is 5; σ_z is roughly 4% of (σ_x, σ_y) and 20% of (τ_{xz}, τ_{yz}), and therefore becomes important for these very thick plate configurations.

Shells incorporate the σ_z approximations with respect to the flat plate, plus an additional accuracy consideration. As Koiter (10) states, the transverse normal stress is of order h/R times the bending stress. For the shell panel:

$$\begin{aligned} \sigma_z &\approx h/R\ (\sigma_x, \sigma_y) \\ \tau_{xz}, \tau_{yz} &\approx h/L\ (\sigma_x, \sigma_y) \end{aligned} \tag{3.7}$$

When combined together, these equations give:

$$\sigma_z \approx L/R\ (\tau_{xz}, \tau_{yz}) \tag{3.8}$$

By the mere fact the maximum h/R ratio used in this thesis is 1/5, σ_z becomes important because it is in reality roughly 20% of the bending stresses. For the smaller h/R ratios used, the σ_z effect is negligible, except for the regions of small a/h as explained above.

Using Eq (3.8), σ_z may be further examined for the h/R ratio of 1/5. L is not always equal to the dimension of the shell panel; it varies with b/h, and is determined from the mode shape in Eq (3.3). As discussed before, the longitudinal mode shape generally behaves like that of a flat plate: usually one full sine wave or one half sine wave. The circumferential mode shape varies, depending on the geometry and on the problem (buckling or vibration).

The panel generally buckles into six sine waves in the

circumferential direction; $L \approx 1/6b$ and $L/R \approx 1/30b$ for $h/R=1/5$. From Eq (3.8), σ_z is roughly 15% of (τ_{xz},τ_{yz}) at $b/h = 5$, where the transverse shear is very prominent. $\sigma_z \approx (\tau_{xz},\tau_{yz})$ at $b/h = 30$, but the transverse shear here is very low; so, the effect is negligible.

For the vibration problem involving $h/R = 1/5$, there is in general only 1 to 1.5 full sine waves in the circumferential direction; $L/R \approx 2/15b$. σ_z becomes important at $b/h=5$ because it is roughly 66% of the transverse shear in a region where the shear is very prominent.

The overall conclusion of this section is σ_z is important for h/R values of 1/5 and a/h (b/h) values of 5, especially for the vibration problem. The overall trends displayed by the data, however, are accurate. Figures 3.3 through 3.10 display logical and consistent trends for these configurations.

Whitney (24) presents a method that includes σ_z effects and would improve the accuracy for these particular geometries. In his model, the transverse displacement w is a linear function of z and has the form: $w(x,y,z) = w_o(x,y) + z\phi(x,y)$, where $w_o(x,y)$ is the mid surface transverse displacement, and $\phi(x,y)$ is an additional degree of freedom that must be included in the constitutive relations and equations and motion. This application would be an interesting follow-on to this thesis.

IV. CONCLUSIONS

A theory applicable to symmetrically laminated anisotropic circular cylindrical shell panels of arbitrary geometries has been developed. The theory includes a through the thickness parabolic transverse shear stress and strain distribution and is valid for $0 \leq h/R \leq 1/5$. Analytical solutions for the fundamental frequencies, critical buckling loads, and the corresponding mode shapes are obtained using the Galerkin technique. Simply supported and clamped boundary conditions were investigated. Based upon the analysis, the following conclusions are presented:

The strain displacement relations are very accurate for $0 \leq h/R \leq 1/10$. The results were verified against the Donnell solutions for $0 \leq h/R \leq 1/50$. Since there is no z/R variation in the transverse shear strains, and since σ_z and ϵ_z are assumed equal to zero, some precision is lost for h/R values of 1/5. However, the generated results show very logical and consistent trends at this h/R limit, and consequently the theory here is assumed to be a very good approximation.

The Galerkin technique proved to be an excellent method for solving the five coupled partial differential equations of motion and boundary conditions. The method converged to exact frequencies very quickly for all geometries. Convergence was slower for the buckling problem, particularly for the clamped

boundary condition at h/R = 1/5 and high b/h ratios. The method still works for these cases, but at the cost of a great deal of computer time. Another benefit of the Galerkin technique is that it may be applied to any desired ply layup and more types of boundary conditions.

Parabolic transverse shear effects were measured up against classical solutions for simply supported flat plates with symmetric cross ply laminates. At a/h = 5 the frequency obtained using transverse shear was 49% lower than the classical frequency. At a/h = 40 the difference was only 1%. For aspect ratios of 3 (a/b = 3), the parabolic shear buckling load was 54% lower than the classical buckling load at b/h = 5 and 0.8% lower at b/h = 50.

There is little difference between the parabolic transverse shear model and the Reissner-Mindlin model for simply supported flat plates. The parabolic model is more accurate than the latter for clamped plates; Mindlin theory overpredicts the frequencies by 15-20% for M=N=2 and by 3-5% for M=N=8. The margin of error decreases as the number of terms for each degree of freedom increases.

Transverse shear effects consistently became negligible as a/h (b/h) approached 40 to 50 for all plate and panel configurations.

Increasing h/R from 0 to 1/5 increased membrane and bending coupling , and drove the frequencies and buckling loads to higher values for both boundary conditions. For both boundary

conditions, the $[\pm 45]_s$ laminates yielded higher frequencies than the $[0/90]_s$ laminates, due to the additional inplane and transverse shear terms in the former. Both $[\pm 45]_s$ and $[0/90]_s$ laminates yielded higher frequencies for clamped boundary conditions than for simply supported boundary conditions.

Buckling loads behaved differently. The $[\pm 45]_s$ laminates yielded higher buckling loads than the $[0/90]_s$ laminate for simply supported boundary conditions, but $[0/90]_s$ buckling loads are higher than $[\pm 45]_s$ buckling loads for clamped boundary conditions. The $[\pm 45]_s$ laminates have slightly higher buckling loads for clamped vs simply supported boundary conditions, except at $h/R = 1/5$ where the buckling loads for both boundary conditions are roughly equal. Finally, $[0/90]_s$ laminates have much higher buckling loads for clamped boundary conditions than simply supported boundary conditions.

Rotary inertia effects were negligible for all panel configurations examined. Fundamental frequencies were only about 0.5% higher with rotary inertia removed.

Appendix A: Transverse Shear and Maximum h/R Assumptions

This appendix explains in detail the assumptions made about the transverse shear strains, γ_{yz} and γ_{xz}, discussed in Chapter II. By examining the work of Dennis (8), a measure of the error introduced by assuming no z/R variation with respect to the shear model is developed. The appendix focuses on γ_{yz}, but it should be understood that similar conclusions for γ_{xz} apply.

This thesis assumes $z/R \approx 0$ for the transverse shear strains. It was shown from Eq (2.2) this resulted in

$$\gamma_{yz} = w,_y - \frac{v}{R} + v,_z \tag{A.1}$$

Substituting the displacement relations in Eq (2.1) into Eq (A.1) gives:

$$\gamma_{yz} = w,_y - \left(1 + \frac{z}{R}\right)\frac{v_o}{R} - \frac{z}{R}\psi_y - \frac{z^2}{R}\phi_2 - \frac{z^3}{R}\theta_2 + \frac{v_o}{R} + \psi_y + 2z\phi_2 + 3z^2\theta_2 \tag{A.2}$$

Since $z/R \approx 0$, Eq (A.2) reduces to:

$$\gamma_{yz} = w,_y + \psi_y + 2z\phi_2 + 3z^2\theta_2 \tag{A.3}$$

Evaluating Eq (A.3) at $z = \pm h/2$ and setting the two equations equal to zero to satisfy the boundary conditions gives:

$$\psi_y + w,_y + h\phi_2 + 3/4h^2\theta_2 = 0$$

$$\psi_y + w,_y - h\phi_2 + 3/4h^2\theta_2 = 0$$

from which the following results:

$$\phi_2 = 0$$

$$\theta_2 = - 4/3h^2(\psi_y + w,_y)$$

By performing similar operations with γ_{xz}, similar expressions are obtained for ϕ_1 and θ_1. By substituting ϕ_2, θ_2, ϕ_1, and θ_1 into Eq (2.1), the displacement field in Eq (2.3) is obtained. Of course, the approximation $z/R \approx 0$ for the transverse shear limits the maximum value of h/R for which the strain displacement relations in Eq (2.2) are valid. By examining reference (8), an accurate value for this maximum h/R limit may be obtained.

Dennis assumed a 4th order displacement field rather than a 3rd order field. For the circular cylindrical shell panel, this field is:

$$u = u_o + z\psi_x + z^2\phi_1 + z^3\gamma_1 + z^4\theta_1$$

$$v = \left[1 + \frac{z}{R}\right]v_o + z\psi_y + z^2\phi_2 + z^3\gamma_2 + z^4\theta_2$$

$$w = w \tag{A.4}$$

Inserting these relations into Eq (2.2) without making the assumption $z/R \approx 0$ gives an exact expression for γ_{yz}:

$$\gamma_{yz} = \frac{1}{1 + \frac{z}{R}}\left[\psi_y + \left[2z - \frac{z^2}{R}\right]\phi_2 + \left[3z^2 - \frac{2z^3}{R}\right]\gamma_2 + \left[4z^3 - \frac{3z^4}{R}\right]\theta_2 + w,_y\right] \tag{A.5}$$

Following the similar procedure, Eq (A.5) is evaluated at $z = \pm h/2$ and set equal to zero. Adding the two resulting equations gives:

$$2h\phi_2 - \frac{h^3}{2R}\gamma_2 + h^3\theta_2 = 0$$

from which:

$$\phi_2 = 0$$

$$\theta_2 = \frac{1}{2R}\gamma_2$$

Inserting these values into Eq (A.5) and again evaluating at $z = \pm h/2$ gives:

$$\gamma_{yz} = 0 = \frac{1}{1 \pm \frac{h}{2R}}\left[w,_y + \psi_y + \frac{3h^2}{4}\left(1 - \frac{h^2}{8R^2}\right)\gamma_2\right] \qquad (A.6)$$

from which:

$$\left(1 - \underline{\frac{h^2}{8R^2}}\right)\gamma_2 = \frac{4}{3h^2}(w,_y - \psi_y) \qquad (A.7)$$

If $h/R = 1/5$, the underlined term in Eq (A.7) = 0.005; therefore,

$$\gamma_2 \approx \frac{4}{3h^2}(w,_y - \psi_y) \text{ for } \frac{h}{R} \le \frac{1}{5}$$

Inserting the values of ϕ_2, θ_2, and γ_2 into Eq (A.4), Dennis obtains exactly the same displacement field as in Eq (2.3). In conclusion, Dennis' transverse shear model is accurate up to $h/R = 1/5$ using the relation:

$$\gamma_{yz} = \frac{1}{1 + \frac{z}{R}}\left[w,_y - \frac{v}{R}\right] + v,_z$$

$$\approx \left[1 - \frac{z}{R}\right]\left[w,_y - \frac{v}{R}\right] + v,_z \qquad (A.8)$$

What is the h/R limit for γ_{yz} used in this thesis, that is Eq (A.1) ? Manipulating Eq (A.8) gives:

$$\gamma_{yz} = w,_y - \frac{v}{R} + v,_z - \frac{z}{R}\left[w,_y - \frac{v}{R}\right]$$

or

$$\gamma_{yz_2} = \gamma_{yz_1} - \frac{z}{R}\left[w,_y - \frac{v}{R}\right] \tag{A.9}$$

where

γ_{yz_2} = the exact value of γ_{yz} in Eq (A.8)

γ_{yz_1} = the approximate value of γ_{yz} in Eq (A.1)

and

$$-\frac{z}{R}\left[w,_y - \frac{v}{R}\right] = \text{the error}$$

Substituting the displacement field in Eq (2.3) into Eq (A.9) gives:

$$\gamma_{yz_2} = \gamma_{yz_1} + \left(\frac{z}{R^2} + \frac{z^2}{R^3}\right)v_o + \left(\frac{z^2}{R^2} - \frac{4z^4}{3h^2R^2}\right)\psi_y - \left(\frac{z}{R} + \frac{4z^4}{3h^2R^2}\right)w,_y$$

For z = + h/2 and h/R = 1/5, the shear becomes:

$$\gamma_{yz_2} = \gamma_{xz_1} + \frac{0.11}{R}v_o + 0.0067\psi_y - 0.103w,_y$$

For a 1.0 in thick laminate and a radius of curvature of 5.0 in, the final result is:

$$\gamma_{yz_2} = \gamma_{xz_1} + 0.022v_o + 0.0067\psi_y - 0.103w,_y$$

This is a small error, especially since this thesis is concerned with small deflection theory with rotations of around 5.0°. Therefore, using an h/R limit of 1/5 is a good approximation for the transverse shear expression in Eq (A.1).

Appendix B: <u>Integration By Parts</u>

This appendix briefly summarizes the approach taken for the integration by parts during the development of the kinetic, strain, and potential energies. This development parallels that presented by Shames and Dym (20).

For two functions, $G(x,y)$ and $H(x,y)$, the following scheme applies for the integration by parts of double integrals:

$$\int_0^b \int_0^a GH,_x dxdy = \int_0^b GH \Big|_{x=0}^{x=a} dy - \int_0^b \int_0^a HG,_x dxdy$$

$$= \int_0^b \Big[G(a,y)H(a,y) - G(0,y)H(0,y)\Big]dy - \int_0^b \int_0^a HG,_x dxdy \tag{B.1}$$

and

$$\int_0^b \int_0^a GH,_y dxdy = \int_0^a GH \Big|_{y=0}^{y=b} dx - \int_0^b \int_0^a HG,_y dxdy$$

$$= \int_0^a \Big[G(x,b)H(x,b) - G(x,0)H(x,0)\Big]dx - \int_0^b \int_0^a HG,_y dxdy \tag{B.2}$$

The line integrals represent the boundary conditions along the edges of the panel. Line integrals of the following form may be reduced further to yield boundary conditions at the four corners of the laminate:

$$\int_0^a GH,_x\Big|_{y=0}^{y=b}dx = \int_0^a\Big[G(x,b)H(x,b),_x - G(x,0)H(x,0),_x\Big]dx$$

$$= GH\Big|_{y=0}^{y=b}\Big|_{x=0}^{x=a} - \int_0^a HG,_x\Big|_{y=0}^{y=b} dx$$

$$= G(a,b)H(a,b) - G(a,0)H(a,0) - G(0,b)H(0,b) +$$

$$G(0,0)H(0,0) - \int_0^a\Big[H(x,b)G(x,b),_x - H(x,0)G(x,0),_x\Big]dx$$

(B.3)

and

$$\int_0^b GH,_y\Big|_{x=0}^{x=a}dy = \int_0^b\Big[G(a,y)H(a,y),_y - G(0,y)H(0,y),_y\Big]dy$$

$$= GH\Big|_{x=0}^{x=a}\Big|_{y=0}^{y=b} - \int_0^b HG,_y\Big|_{x=0}^{x=a} dy$$

$$= G(a,b)H(a,b) - G(a,0)H(a,0) - G(0,b)H(0,b) +$$

$$G(0,0)H(0,0) - \int_0^b\Big[H(a,y)G(a,y),_y - H(0,y)G(0,y),_y\Big]dy$$

(B.4)

Appendix C: Integration Formulas used in Generating Galerkin Equations

MACSYMA (25) was used to integrate the equations of motion and boundary conditions to generate the Galerkin equations for the simply supported and clamped boundary conditions. Intrinsic in MACSYMA's artificial intelligence logic is the capability to symbolically integrate trigonometric functions.

MACSYMA evaluated the following integrals, taken from (2), to generate the Galerkin equations:

When m = p:

$$\int_0^a \cos(m\pi x/a)\cos(p\pi x/a)dx = a/2 \tag{C.1}$$

$$\int_0^a \sin(m\pi x/a)\sin(p\pi x/a)dx = a/2 \tag{C.2}$$

$$\int_0^a \sin(m\pi x/a)\cos(p\pi x/a)dx = 0 \tag{C.3}$$

When $m \neq p$:

$$\int_0^a \cos(m\pi x/a)\cos(p\pi x/a)dx = 0 \tag{C.4}$$

$$\int_0^a \sin(m\pi x/a)\sin(p\pi x/a)dx = 0 \tag{C.5}$$

$$\int_0^a \sin(m\pi x/a)\cos(p\pi x/a)dx$$

$$= \begin{cases} 0 \text{ for } (m+p) \text{ an even integer} \\ \dfrac{2am}{\pi(m^2 - p^2)} \text{ for } (m+p) \text{ an odd integer} \end{cases} \tag{C.6}$$

Similarly, for

$$\int_0^a \sin(p\pi x/a)\cos(m\pi x/a)dx$$

$$= \begin{cases} 0 \text{ for } (m+p) \text{ an even integer} \\ \dfrac{2ap}{\pi(p^2 - m^2)} \text{ for } (m+p) \text{ an odd integer} \end{cases} \tag{C.7}$$

These same integrals apply for n and q, when the independent variable is y, and the integration is from 0 to b.

Appendix D: Computer Programs

This appendix contains the computer listings. Program "MAINTHESIS", the complete code for simply supported boundary conditions, is listed first. This is followed by the code used for clamped boundary conditions.

****** Simply Supported Boundary Condition ******

```
      PROGRAM MAINTHESIS
C-------------------------------------------------------------------
C     CAPT PETE LINNEMANN
C
C     GA-88D
C
C     THE DETERMINATION OF THE FUNDAMENTAL NATURAL FREQUENCY AND
C     CRITICAL BUCKLING LOAD OF AN ANISOTROPIC LAMINATED CIRCULAR
C     CYLINDRICAL SHELL PANEL INCLUDING THE EFFECTS OF PARABOLIC
C     TRANSVERSE SHEAR DEFORMATION AND ROTARY INERTIA.
C
C     THESIS ADVISOR:  DR ANTHONY PALAZATTO
C
C     INITIAL PROGRAMMING DATE:  26 JUL 88
C
C-------------------------------------------------------------------
C
C     INITIALIZATION
C
      DOUBLE PRECISION A,B,R,H,PI,A11,A12,A22,A16,A26,A66,A44,A45,
     1A55,D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26,
     2F66,F44,F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26,
     3J66,TPLY,THETA(100),E1,E2,G12,V12,V21,G13,G23,STIFF(500,500)
     4,MASS(500,500),BETA(500),RHO,REVEC(100)
C
      DOUBLE COMPLEX ALPHA(500),EVAL(500),EVEC(500,500)
C
C     WORKSPACE ALLOCATION FOR IMSL
C
      COMMON / WORKSP / RWKSP
      REAL RWKSP(1503026)
      CALL IWKIN(1503026)
C
      OPEN (UNIT = 1, FILE = 'MAININ.', STATUS = 'OLD')
      OPEN (UNIT = 2, FILE = 'MAINOUT.', STATUS = 'NEW')
C
C     IS THIS A VIBRATION PROBLEM OR A BUCKLING PROBLEM ?
C     NBUCVIB = 1;  VIBRATION.     NBUCVIB = 2;  BUCKLING.
C
      READ(1,5) NBUCVIB
C
C-------------------------------------------------------------------
C
C     READ SHELL PANEL DIMENSIONS AND LAMINATE DATA
C
C     DIMENSIONAL DATA
C
```

```
C     LENGTH IN THE X DIRECTION (LONGITUDINAL AXIS)
      READ(1,1) A
C     LENGTH IN THE Y DIRECTION (CIRCUMFERENCIAL AXIS)
      READ(1,1) B
C     RADIUS OF CURVATURE
      READ(1,1) R
C     LAMINATE THICKNESS
      READ(1,1) H
      PI = 3.141592653589793
C     LENGTH TO SPAN RATIO AND THICKNESS RATIO
      AOVERB = A/B
      HOVERR = H/R
      AOVERH = A/H
      BOVERH = B/H
C
C     LAMINATE DATA
C
C     NUMBER OF PLYS IN THE LAMINATE
      READ(1,5) NPLYS
C     THICKNESS OF EACH PLY IN THE LAMINATE
      TPLY = H / NPLYS
C     ORIENTATION ANGLE OF EACH PLY IN THE LAMINATE
      DO 100 I = 1,NPLYS
      READ(1,1) THETA(I)
100   CONTINUE
C     MATERIAL PROPERTIES OF EACH PLY
      READ(1,1) E1
      READ(1,1) E2
      READ(1,1) G12
      READ(1,1) V12
      READ(1,3) RHO
      V21 = V12 * E2 / E1
C     FOR THIS THESIS, G13 AND G23 WILL HAVE THE FOLLOWING VALUES:
      G13 = G12
      G23 = 0.8 * G12
C
C--------------------------------------------------------------
C
C     WRITE SHELL PANEL DIMENSIONS AND LAMINATE DATA
C
      WRITE(2,10)
      IF(NBUCVIB .EQ. 1) THEN
      WRITE (2,11)
      ELSE
      WRITE (2,12)
      ENDIF
      WRITE(2,13)
      WRITE(2,15)
      WRITE(2,17) A,B,AOVERB
      WRITE(2,18) H,AOVERH,BOVERH
      WRITE(2,20) R,HOVERR
      WRITE(2,22)
```

```
      WRITE(2,23)
      DO 101 I = 1, NPLYS
      WRITE(2,24) THETA(I)
  101 CONTINUE
      WRITE(2,25) NPLYS,H
      WRITE(2,26) TPLY
      WRITE(2,28) E1,E2
      WRITE(2,30) G12
      WRITE(2,32) G13,G23
      WRITE(2,34) V12,V21
      WRITE(2,35) RHO
C
C---------------------------------------------------------------
C
C     CALCULATE THE BENDING, EXTENSIONAL, AND HIGHER ORDER
C     STIFFNESS ELEMENTS FOR A SYMMETRIC LAMINATE.
C
      CALL LAMINAT(NPLYS,TPLY,THETA,E1,E2,G12,V12,
     1V21,G13,G23,PI,H,A11,A12,A22,A16,A26,A66,A44,A45,A55,
     2D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26,F66,
     3F44,F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26,J66)
C
C---------------------------------------------------------------
C
C     WRITE LAMINATE STIFFNESS ELEMENTS
C
      WRITE(2,36)
      WRITE(2,40) A11,A12,A22
      WRITE(2,41) A16,A26,A66
      WRITE(2,42) A44,A45,A55
      WRITE(2,43)
      WRITE(2,44) D11,D12,D22
      WRITE(2,45) D16,D26,D66
      WRITE(2,46) D44,D45,D55
      WRITE(2,47)
      WRITE(2,48)
      WRITE(2,49) F11,F12,F22
      WRITE(2,50) F16,F26,F66
      WRITE(2,51) F44,F45,F55
      WRITE(2,52) H11,H12,H22
      WRITE(2,53) H16,H26,H66
      WRITE(2,54) J11,J12,J22
      WRITE(2,55) J16,J26,J66
C
C---------------------------------------------------------------
C     READ THE NUMBER OF TERMS IN THE ADMISSIBLE FUNCTIONS.
C     DETERMINE THE DIMENSION OF THE MASS AND STIFFNESS MATRICES.
C
      READ(1,5) MMAX
      MSIZE = 5 * MMAX * MMAX
      MSIZESQ = MMAX * MMAX
      IF (NBUCVIB .EQ. 1) THEN
```

```
      WRITE(2,57)
      WRITE(2,59) MMAX,MSIZE,MSIZE
      WRITE(2,60)
      ELSE
      WRITE(2,58)
      WRITE(2,63)
      WRITE(2,64)
      WRITE(2,59) MMAX,MSIZE,MSIZE
      ENDIF
C-------------------------------------------------------------
C
C     USING THE BENDING, EXTENSIONAL, AND HIGHER ORDER STIFFNESS
C     ELEMENTS AND THE SHELL PANEL PHYSICAL CHARACTERISTICS AS
C     INPUTS, CALCULATE THE STIFFNESS AND MASS MATRICES AND THEN
C     FIND THE NATURAL FREQUENCIES AND/OR AXIAL BUCKLING LOAD AND
C     THEIR RESPECTIVE MODE SHAPES.
C
      CALL GALERK(PI,R,H,A,B,A11,A12,A22,A16,A26,A66,A44,A45,A55,
     1D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26,F66,
     2F44,F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26,J66,
     3NBUCVIB,MMAX,MSIZE,RHO,STIFF,MASS,BETA,ALPHA,EVAL,EVEC,
     4MSIZESQ,REVEC)
C-------------------------------------------------------------
      STOP
C
C-------------------------------------------------------------
C
C     FORMAT STATEMENTS
C
    1 FORMAT(F15.5)
    3 FORMAT(D22.15)
    4 FORMAT(E12.5)
    5 FORMAT(I5)
   10 FORMAT(////,5X,'ANISOTROPIC LAMINATED CIRCULAR CYLINDRICAL S
     *HELL PANEL')
   11 FORMAT(//,5X,'VIBRATION PROBLEM')
   12 FORMAT(//,5X,'BUCKLING PROBLEM')
   13 FORMAT(//,5X,'S2 SIMPLY SUPPORTED BOUNDARY CONDITIONS')
   15 FORMAT(///,5X,'SHELL PANEL DIMENSIONS (in.)')
   17 FORMAT(/,5X,'a = ',1X,F6.2,4X,'b = ',1X,F6.2,4X,'a/b = ',1X,
     *F6.2)
   18 FORMAT(/,5X,'h = ',1X,F4.2,4X,'a/h = ',1X,F6.2,4X,'b/h = ',1
     *X,F6.2)
   20 FORMAT(/,5X,'R = ',1X,E12.5,6X,'h/R = ',1X,F10.8)
   22 FORMAT(//,5X,'SHELL PANEL LAMINATE DIMENSIONAL AND MATERIAL
     *DATA')
   23 FORMAT(//,5X,'SYMMETRIC LAMINATE PLY LAYUP (DEGREES)')
   24 FORMAT(/,30X,F7.2)
   25 FORMAT(/,3X,I3,2X,'PLYS IN THIS',2X,F4.2,2X,'in. THICK LAMIN
     *ATE')
   26 FORMAT(/,5X,'EACH PLY IS',1X,E12.5,2X,'ins. THICK')
   28 FORMAT(/,5X,'ELASTICITY MODULII (psi):  E1 = ',E12.5,2X,'E2
```

```
     *= ',E12.5)
   30 FORMAT(/,5X,'IN PLANE SHEAR MODULUS (psi):  G12 = ',E12.5)
   32 FORMAT(/,5X,'TRANSVERSE SHEAR MODULII (psi):  G13 = ',E12.5,
     *2X,'G23 = ',E12.5)
   34 FORMAT(/,5X,'POISSONS RATIOS:  V12 = ',1X,F6.4,3X,'V21 = ',1
     *X,F6.4)
   35 FORMAT(/,5X,'MASS DENSITY (LB*SEC^2/IN^4):  RHO = ',1X,D18.1
     *1)
   36 FORMAT(///,5X,'EXTENSIONAL STIFFNESS ELEMENTS (lb/in)')
   40 FORMAT(/,5X,'A11 =',F15.3,3X,'A12 =',F15.3,3X,'A22 =',F15.3)
   41 FORMAT(/,5X,'A16 =',F15.3,3X,'A26 =',F15.3,3X,'A66 =',F15.3)
   42 FORMAT(/,5X,'A44 =',F15.3,3X,'A45 =',F15.3,3X,'A55 =',F15.3)
   43 FORMAT(//,5X,'BENDING STIFFNESS ELEMENTS (lb * in)')
   44 FORMAT(/,5X,'D11 =',F15.3,3X,'D12 =',F15.3,3X,'D22 =',F15.3)
   45 FORMAT(/,5X,'D16 =',F15.3,3X,'D26 =',F15.3,3X,'D66 =',F15.3)
   46 FORMAT(/,5X,'D44 =',F15.3,3X,'D45 =',F15.3,3X,'D55 =',F15.3)
   47 FORMAT(//,5X,'HIGHER ORDER STIFFNESS ELEMENTS')
   48 FORMAT(5X,'Fij (in * lb^3),Hij (in * lb^5),Jij (in * lb^7)')
   49 FORMAT(//,5X,'F11 =',F15.3,3X,'F12 =',F15.3,3X,'F22=',F15.3)
   50 FORMAT(/,5X,'F16 =',F15.3,3X,'F26 =',F15.3,3X,'F66 =',F15.3)
   51 FORMAT(/,5X,'F44 =',F15.3,3X,'F45 =',F15.3,3X,'F55 =',F15.3)
   52 FORMAT(//,5X,'H11 =',F15.3,3X,'H12 =',F15.3,3X,'H22=',F15.3)
   53 FORMAT(/,5X,'H16 =',F15.3,3X,'H26 =',F15.3,3X,'H66 =',F15.3)
   54 FORMAT(//,5X,'J11 =',F15.3,3X,'J12 =',F15.3,3X,'J22=',F15.3)
   55 FORMAT(/,5X,'J16 =',F15.3,3X,'J26 =',F15.3,3X,'J66 =',F15.3)
   57 FORMAT(///,5X,'VIBRATION EIGENVALUE ANALYSIS - FIRST 10 MODE
     *S PRINTED')
   58 FORMAT(///,5X,'BUCKLING EIGENVALUE ANALYSIS - ALL MODES PRIN
     *TED')
   59 FORMAT(//,5X,'MMAX = NMAX =   ',I2,5X,'STIFFNESS AND MASS/INE
     *RTIA MATRICES ARE (',I3,1X,'BY',1X,I3,')')
   60 FORMAT(///,5X,'MODE NUMBER',11X,'EIGENVALUE',14X,'NATURAL FR
     *EQUENCY (RAD/SEC)')
   63 FORMAT(5X,'THE CRITICAL BUCKLING LOAD IS THE EIGENVALUE WITH
     *')
   64 FORMAT(5X,'THE SMALLEST ABSOLUTE VALUE')
C-----------------------------------------------------------------
      END
C-----------------------------------------------------------------
      SUBROUTINE LAMINAT(NPLYS,TPLY,THETA,E1,E2,G12,V12,
     1V21,G13,G23,PI,H,A11,A12,A22,A16,A26,A66,A44,A45,A55,
     2D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26,F66,
     3F44,F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26,J66)
C
C-----------------------------------------------------------------
C
C     THIS SUBROUTINE CALCULATES THE BENDING, EXTENSIONAL, AND
C     HIGHER ORDER STIFFNESS ELEMENTS FOR THE LAMINATE.
C
C     THE LAMINATE IS SYMMETRIC WITH RESPECT TO THE MIDPLANE IN
C     BOTH MATERIAL PROPERTIES AND ORIENTATION ANGLE, THETA.
C     THIS THESIS ASSUMES A HOMOGENEOUS LAMINATE -- MATERIAL
```

```
C   PROPERTIES ARE IDENTICAL FOR EACH PLY.  THE ONLY THING THAT
C   CAN CHANGE IS ORIENTATION ANGLE.
C   EACH PLY ALSO HAS THE SAME THICKNESS.
C
C-------------------------------------------------------------------
C
      DOUBLE PRECISION H,PI,A11,A12,A22,A16,A26,A66,A44,A45,
     1A55,D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26,
     2F66,F44,F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26,
     3J66,TPLY,THETA(100),E1,E2,G12,V12,V21,G13,G23,Q11,Q12,Q22,
     4Q44,Q55,Q66,QBAR11,QBAR12,QBAR16,QBAR22,QBAR26,QBAR44,QBAR45
     5,QBAR55,QBAR66,ZK,ZK1,TH(100),ZK0,ZK3,ZK5,ZK7,ZK9
C
C   REDUCED STIFFNESS ELEMENTS IN PRINCIPLE COORDINATES
      Q11 = E1 / (1.0 - V12 * V21)
      Q12 = V12 * E2 /.(1.0 - V12 * V21)
      Q22 = E2 / (1.0 - V12 * V21)
      Q44 = G23
      Q55 = G13
      Q66 = G12
C   INITIALIZE ALL STIFFNESS ELEMENTS TO ZERO
      A11 = 0.
      A12 = 0.
      A22 = 0.
      A16 = 0.
      A26 = 0.
      A66 = 0.
      A44 = 0.
      A45 = 0.
      A55 = 0.
      D11 = 0.
      D12 = 0.
      D22 = 0.
      D16 = 0.
      D26 = 0.
      D66 = 0.
      D44 = 0.
      D45 = 0.
      D55 = 0.
      F11 = 0.
      F12 = 0.
      F22 = 0.
      F16 = 0.
      F26 = 0.
      F66 = 0.
      F44 = 0.
      F45 = 0.
      F55 = 0.
      H11 = 0.
      H12 = 0.
      H22 = 0.
      H16 = 0.
```

```
      H26 = 0.
      H66 = 0.
      J11 = 0.
      J12 = 0.
      J22 = 0.
      J16 = 0.
      J26 = 0.
      J66 = 0.
C-----------------------------------------------------------------------
C     IN ORDER FROM THE FIRST PLY AT Z = - H/2 TO THE TOP PLY AT
C     Z = + H/2, INPUT THE PLY ORIENTATION ANGLE, THETA.  THEN IN
C     TURN CALCULATE THE QBARS AND THE STIFFNESS ELEMENTS FOR THAT
C     PLY.  REPEAT THE PROCEDURE FOR ALL PLYS, THEN ADD THE PLY
C     STIFFNESS ELEMENTS TOGETHER TO GET THE LAMINATE STIFFNESS
C     ELEMENTS.
C     INITIALIZE Z TO THE BOTTOM OF THE LAMINATE
C-----------------------------------------------------------------------
      ZK1 = - H / 2.0
      DO 100 I = 1,NPLYS
      TH(I) = THETA(I) * PI / 180.0
C-----------------------------------------------------------------------
C     COMPUTE THE QBARS - TRANSFORMED REDUCED STIFFNESS ELEMENTS
C     IN GLOBAL COORDINATES.
C-----------------------------------------------------------------------
      QBAR11 = Q11 * DCOS(TH(I))**4 + 2.0 * (Q12 + 2.0 * Q66) *
     1DSIN(TH(I))**2 * DCOS(TH(I))**2 + Q22 * DSIN(TH(I))**4
      QBAR12 = (Q11 + Q22 - 4.0 * Q66) * DSIN(TH(I))**2 *
     1DCOS(TH(I))**2 + Q12 * (DSIN(TH(I))**4 + DCOS(TH(I))**4)
      QBAR16 = (Q11 - Q12 - 2.0 * Q66) * DSIN(TH(I)) * DCOS(TH(I))
     1**3 + (Q12 - Q22 + 2.0 * Q66) * DSIN(TH(I))**3 * DCOS(TH(I))
      QBAR22 = Q11 * DSIN(TH(I))**4 + 2.0 * (Q12 + 2.0 * Q66) *
     1DSIN(TH(I))**2 * DCOS(TH(I))**2 + Q22 * DCOS(TH(I))**4
      QBAR26 = (Q11 - Q12 - 2.0 * Q66) * DSIN(TH(I))**3 * DCOS(TH
     1(I))+ (Q12 - Q22 + 2.0 * Q66) * DSIN(TH(I)) * DCOS(TH(I))**3
      QBAR44 = Q44 * DCOS(TH(I))**2 + Q55 * DSIN(TH(I))**2
      QBAR45 = (Q44 - Q55) * DCOS(TH(I)) * DSIN(TH(I))
      QBAR55 = Q55 * DCOS(TH(I))**2 + Q44 * DSIN(TH(I))**2
      QBAR66 = (Q11 + Q22 - 2.0 * Q12 - 2.0 * Q66) *DSIN(TH(I))**2
     1* DCOS(TH(I))**2 + Q66 * (DSIN(TH(I))**4 + DCOS(TH(I))**4)
C-----------------------------------------------------------------------
C     TOP AND BOTTOM LOCATION OF PLY(I)
      ZK = ZK1 + TPLY
C     EXTENSIONAL STIFFNESS ELEMENTS
      ZKO = ZK - ZK1
      A11 = QBAR11 * ZKO + A11
      A12 = QBAR12 * ZKO + A12
      A22 = QBAR22 * ZKO + A22
      A16 = QBAR16 * ZKO + A16
      A26 = QBAR26 * ZKO + A26
      A66 = QBAR66 * ZKO + A66
      A44 = QBAR44 * ZKO + A44
      A45 = QBAR45 * ZKO + A45
```

```
      A55 = QBAR55 * ZK0 + A55
C     BENDING STIFFNESS ELEMENTS
      ZK3 = (ZK**3 - ZK1**3) / 3.0
      D11 = QBAR11 * ZK3 + D11
      D12 = QBAR12 * ZK3 + D12
      D22 = QBAR22 * ZK3 + D22
      D16 = QBAR16 * ZK3 + D16
      D26 = QBAR26 * ZK3 + D26
      D66 = QBAR66 * ZK3 + D66
      D44 = QBAR44 * ZK3 + D44
      D45 = QBAR45 * ZK3 + D45
      D55 = QBAR55 * ZK3 + D55
C     HIGHER ORDER STIFFNESS ELEMENTS
      ZK5 = (ZK**5 - ZK1**5) / 5.0
      F11 = QBAR11 * ZK5 + F11
      F12 = QBAR12 * ZK5 + F12
      F22 = QBAR22 * ZK5 + F22
      F16 = QBAR16 * ZK5 + F16
      F26 = QBAR26 * ZK5 + F26
      F66 = QBAR66 * ZK5 + F66
      F44 = QBAR44 * ZK5 + F44
      F45 = QBAR45 * ZK5 + F45
      F55 = QBAR55 * ZK5 + F55
      ZK7 = (ZK**7 - ZK1**7) / 7.0
      H11 = QBAR11 * ZK7 + H11
      H12 = QBAR12 * ZK7 + H12
      H22 = QBAR22 * ZK7 + H22
      H16 = QBAR16 * ZK7 + H16
      H26 = QBAR26 * ZK7 + H26
      H66 = QBAR66 * ZK7 + H66
      ZK9 = (ZK**9 - ZK1**9) / 9.0
      J11 = QBAR11 * ZK9 + J11
      J12 = QBAR12 * ZK9 + J12
      J22 = QBAR22 * ZK9 + J22
      J16 = QBAR16 * ZK9 + J16
      J26 = QBAR26 * ZK9 + J26
      J66 = QBAR66 * ZK9 + J66
C     GO TO NEXT LAYER
      ZK1 = ZK
100   CONTINUE
      RETURN
      END
C-----------------------------------------------------------------------
      SUBROUTINE GALERK(PI,R,H,A,B,A11,A12,A22,A16,A26,A66,A44,A45
     1,A55,D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26
     2,F66,F44,F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26
     3,J66,NBUCVIB,MMAX,MSIZE,RHO,STIFF,MASS,BETA,ALPHA,EVAL,EVEC,
     4MSIZESQ,REVEC)
C-----------------------------------------------------------------------
C     THIS SUBROUTINE GENERATES THE GALERKIN EQUATIONS AND FORMS
C     THE MASS AND STIFFNESS MATRICES.  THEN IT CALLS DGVCRG, AN
C     IMSL SUBROUTINE WHICH SOLVES THE EIGENVALUE PROBLEM:
```

```
C           [STIFF]{X} = (OMEGA^2 OR N1BAR)[MASS]{X}.
C-----------------------------------------------------------------
     DOUBLE PRECISION  PI,R,H,A,B,A11,A12,A22,A16,A26,A66,A44,
    1A45,A55,D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,
    2F26,F66,F44,F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,
    3J26,J66,STIFF(MSIZE,MSIZE),MASS(MSIZE,MSIZE),AUO,BUO,CUO,EUO
    4,GUO,AVO,BVO,CVO,EVO,GVO,AW,BW,CW,EW,GW,AJX,BJX,CJX,EJX,GJX,
    5AJY,BJY,CJY,EJY,GJY,AUOMASS,BUOMASS,CUOMASS,EUOMASS,GUOMASS,
    6AVOMASS,BVOMASS,CVOMASS,EVOMASS,GVOMASS,AWMASS,BWMASS,CWMASS
    7,EWMASS,GWMASS,AJXMASS,BJXMASS,CJXMASS,EJXMASS,GJXMASS,AJYMA
    8SS,BJYMASS,CJYMASS,EJYMASS,GJYMASS,RHO,I2BARPR,I3BARPR,I5BAR
    9,I7,I1,I4BAR
     INTEGER P,Q
C    THESE VARIABLES NEEDED FOR THE IMSL EIGENVALUE SOLVER.
     DOUBLE PRECISION BETA(MSIZE),REVAL,OMEGA,AGEVAL,AGEVEC,
    1REVEC(MSIZESQ)
     DOUBLE COMPLEX ALPHA(MSIZE),EVAL(MSIZE),EVEC(MSIZE,MSIZE)
C-----------------------------------------------------------------
C    NUMBER OF TERMS IN THE ADMISSIBLE FUNCTIONS
     NMAX = MMAX
C    GENERATE GALERKIN EQUATIONS
     I = 1
     J = 1
     DO 10 P = 1,MMAX
     DO 10 Q = 1,NMAX
     DO 20 M = 1,MMAX
     DO 20 N = 1,NMAX
C-----------------------------------------------------------------
C    COMPUTE STIFFNESS MATRIX ELEMENTS
C-----------------------------------------------------------------
     IF (M .EQ. P .AND. N .EQ. Q) THEN
C
     AUO =
    1(3*PI**2*A*D66*H**2-4*PI**2*A*F66)*Q**2/(B*H**2*R)/8.0
     BUO =
    1((3*PI**2*D66+6*PI**2*D12)*H**2-4*PI**2*F66-8*PI**2*F12)*P*Q
    1   /(H**2*R)/24.0
     CUO =
    1-((8*PI**3*F66+4*PI**3*F12)*P*Q**2-3*PI*A12*B**2*H**2*P)/(B*
    1H**2*R)/12.0
     EUO =
    1-((4*PI**2*A**2*A66*Q**2+4*PI**2*A11*B**2*P**2)*R**2+PI**2*A
    1**2*D66*Q**2)/(A*B*R**2)/16.0
     GUO =
    1-((4*PI**2*A66+4*PI**2*A12)*P*Q*R**2-PI**2*D66*P*Q)/R**2/16.
     AVO =
    1(3*PI**2*D66*H**2-4*PI**2*F66)*P*Q/(H**2*R)/24.0
     BVO =
    1((6*PI**2*A**2*D22*H**2-8*PI**2*A**2*F22)*Q**2+(4*PI**2*B**2
    1*F66-3*PI**2*B**2*D66*H**2)*P**2)/(A*B*H**2*R)/24.0
     CVO =
    1-(4*PI**3*A*F22*Q**3-3*PI*A*A22*B**2*H**2*Q)/(B**2*H**2*R)/1
```

```
12.0
 EVO =
1-((4*PI**2*A66+4*PI**2*A12)*P*Q*R**2-PI**2*D66*P*Q)/R**2/16.
 GVO =
1-((4*PI**2*A**2*A22*Q**2+4*PI**2*A66*B**2*P**2)*R**2+PI**2*B
1**2*D66*P**2)/(A*B*R**2)/16.0
 AW =
1-(((32*PI**3*A**2*H66+16*PI**3*A**2*H12+(-24*PI**3*A**2*F66-
112*PI**3*A**2*F12)*H**2)*P*Q**2+(16*PI**3*B**2*H11-12*PI**3*
1B**2*F11*H**2)*P**3+(9*PI*A**2*A55*B**2*H**4-72*PI*A**2*B**2
1*D515*H**2+144*PI*A**2*B**2*F55)*P)*R**2+(16*PI**3*A**2*J66-
112*PI**3*A**2*H**2*H66)*P*Q**2)/(A**2*B*H**4*R**2)/36.0
 BW =
1-((((16*PI**3*A**2*H22-12*PI**3*A**2*F22*H**2)*Q**3+((32*PI**
13*B**2*H66+16*PI**3*B**2*H12+(-24*PI**3*B**2*F66-12*PI**3*B*
1*2*F12)*H**2)*P**2+9*PI*A**2*A44*B**2*H**4-72*PI*A**2*B**2*D
144*H**2+144*PI*A**2*B**2*F44)*Q)*R**2+(16*PI**3*A**2*J22-12*
1PI**3*A**2*H**2*H22)*Q**3+(9*PI*A**2*B**2*D22*H**4-12*PI*A**
12*B**2*F22*H**2)*Q)/(A*B**2*H**4*R**2)/36.0
 CW =
1-((16*PI**4*A**4*H22*Q**4+((64*PI**4*A**2*B**2*H66+32*PI**4*
1A**2*B**2*H12)*P**2+9*PI**2*A**4*A44*B**2*H**4-72*PI**2*A**4
1*B**2*D44*H**2+144*PI**2*A**4*B**2*F44)*Q**2+16*PI**4*B**4*H
111*P**4+(9*PI**2*A**2*A55*B**4*H**4-72*PI**2*A**2*B**4*D55*H
1**2+144*PI**2*A**2*B**4*F55)*P**2)*R**2+16*PI**4*A**4*J22*Q*
1*4+(16*PI**4*A**2*B**2*J66*P**2-24*PI**2*A**4*B**2*F22*H**2)
1*Q**2+9*A**4*A22*B**4*H**4)/(A**3*B**3*H**4*R**2)/36.0
 EW =
1-((8*PI**3*F66+4*PI**3*F12)*P*Q**2-3*PI*A12*B**2*H**2*P)/(B*
1H**2*R)/12.0
 GW =
1-(4*PI**3*A*F22*Q**3-3*PI*A*A22*B**2*H**2*Q)/(B**2*H**2*R)/1
12.0
 AJX =
1-((((16*PI**2*A**2*H66+9*PI**2*A**2*D66*H**4-24*PI**2*A**2*F6
16*H**2)*Q**2+(16*PI**2*B**2*H11+9*PI**2*B**2*D11*H**4-24*PI*
1*2*B**2*F11*H**2)*P**2+9*A**2*A55*B**2*H**4-72*A**2*B**2*D55
1*H**2+144*A**2*B**2*F55)*R**2+(16*PI**2*A**2*J66-24*PI**2*A*
1*2*H**2*H66+9*PI**2*A**2*F66*H**4)*Q**2)/(A*B*H**4*R**2)/36.
 BJX =
1-(16*PI**2*H66+16*PI**2*H12+(9*PI**2*D66+9*PI**2*D12)*H**4+(
1   -24*PI**2*F66-24*PI**2*F12)*H**2)*P*Q/H**4/36.0
 CJX =
1-(((32*PI**3*A**2*H66+16*PI**3*A**2*H12+(-24*PI**3*A**2*F66-
112*PI**3*A**2*F12)*H**2)*P*Q**2+(16*PI**3*B**2*H11-12*PI**3*
1B**2*F11*H**2)*P**3+(9*PI*A**2*A55*B**2*H**4-72*PI*A**2*B**2
1*D55*H**2+144*PI*A**2*B**2*F55)*P)*R**2+(16*PI**3*A**2*J66-1
12*PI**3*A**2*H**2*H66)*P*Q**2)/(A**2*B*H**4*R**2)/36.0
 EJX =
1(3*PI**2*A*D66*H**2-4*PI**2*A*F66)*Q**2/(B*H**2*R)/8.0
 GJX =
1(3*PI**2*D66*H**2-4*PI**2*F66)*P*Q/(H**2*R)/24.0
```

```
      AJY =
     1-(16*PI**2*H66+16*PI**2*H12+(9*PI**2*D66+9*PI**2*D12)*H**4+(
     1   -24*PI**2*F66-24*PI**2*F12)*H**2)*P*Q/H**4/36.0
      BJY =
     1-(((16*PI**2*A**2*H22+9*PI**2*A**2*D22*H**4-24*PI**2*A**2*F2
     12*H**2)*Q**2+(16*PI**2*B**2*H66+9*PI**2*B**2*D66*H**4-24*PI*
     1*2*B**2*F66*H**2)*P**2+9*A**2*A44*B**2*H**4-72*A**2*B**2*D44
     1*H**2+144*A**2*B**2*F44)*R**2+(16*PI**2*A**2*J22-24*PI**2*A*
     1*2*H**2*H22+9*PI**2*A**2*F22*H**4)*Q**2)/(A*B*H**4*R**2)/36.
      CJY =
     1-(((16*PI**3*A**2*H22-12*PI**3*A**2*F22*H**2)*Q**3+((32*PI**
     13*B**2*H66+16*PI**3*B**2*H12+(-24*PI**3*B**2*F66-12*PI**3*B*
     1*2*F12)*H**2)*P**2+9*PI*A**2*A44*B**2*H**4-72*PI*A**2*B**2*D
     144*H**2+144*PI*A**2*B**2*F44)*Q)*R**2+(16*PI**3*A**2*J22-12*
     1PI**3*A**2*H**2*H22)*Q**3+(9*PI*A**2*B**2*D22*H**4-12*PI*A**
     12*B**2*F22*H**2)*Q)/(A*B**2*H**4*R**2)/36.0
      EJY =
     1((3*PI**2*D66+6*PI**2*D12)*H**2-4*PI**2*F66-8*PI**2*F12)*P*Q
     1   /(H**2*R)/24.0
      GJY =
     1((6*PI**2*A**2*D22*H**2-8*PI**2*A**2*F22)*Q**2+(4*PI**2*B**2
     1*F66-3*PI**2*B**2*D66*H**2)*P**2)/(A*B*H**2*R)/24.0
C
      ELSEIF (MOD(M + P,2) .NE. 0 .AND. MOD(N + Q,2) .NE. 0) THEN
C
      AUO =
     1-((12*D16*H**2-16*F16)*N*P**2+(6*D16*H**2-8*F16)*M**2*N)*Q/(
     1((3*H**2*P**2-3*H**2*M**2)*Q**2-3*H**2*N**2*P**2+3*H**2*M**2
     1*N**2)*R)
      BUO =
     1-(6*A*D26*H**2-8*A*F26)*M*N**2*Q/(((B*H**2*P**2-B*H**2*M**2)
     1*Q**2-B*H**2*N**2*P**2+B*H**2*M**2*N**2)*R)
      CUO =
     1(16*PI**2*B**2*F16*M*N*P**2+24*PI**2*A**2*F26*M*N**3+(8*PI**
     12*B**2*F16*M**3-12*A**2*A26*B**2*H**2*M)*N)*Q/(((3*PI*A*B**2
     1*H**2*P**2-3*PI*A*B**2*H**2*M**2)*Q**2-3*PI*A*B**2*H**2*N**2
     1*P**2+3*PI*A*B**2*H**2*M**2*N**2)*R)
      EUO =
     1(4*A16*N*P**2+4*A16*M**2*N)*Q/((P**2-M**2)*Q**2-N**2*P**2+M*
     1*2*N**2)
      GUO =
     1(4*A16*B**2*M*P**2+4*A**2*A26*M*N**2)*Q/((A*B*P**2-A*B*M**2)
     1*Q**2-A*B*N**2*P**2+A*B*M**2*N**2)
      AVO =
     1-((12*A**2*D26*H**2-16*A**2*F26)*N*P*Q**2+(8*B**2*F16-6*B**2
     1*D16*H**2)*M**2*N*P)/(((3*A*B*H**2*P**2-3*A*B*H**2*M**2)*Q**
     12-3*A*B*H**2*N**2*P**2+3*A*B*H**2*M**2*N**2)*R)
      BVO =
     1-(6*D26*H**2-8*F26)*M*N**2*P/(((3*H**2*P**2-3*H**2*M**2)*Q**
     12-3*H**2*N**2*P**2+3*H**2*M**2*N**2)*R)
      CVO =
     1(16*PI**2*A**2*F26*M*N*P*Q**2+(8*PI**2*A**2*F26*M*N**3+(-8*P
```

```
1I**2*B**2*F16*M**3-12*A**2*A26*B**2*H**2*M)*N)*P)/(((3*PI*A*
1*2*B*H**2*P**2-3*PI*A**2*B*H**2*M**2)*Q**2-3*PI*A**2*B*H**2*
1N**2*P**2+3*PI*A**2*B*H**2*M**2*N**2)*R)
 EVO =
1(4*A**2*A26*N*P*Q**2+4*A16*B**2*M**2*N*P)/((A*B*P**2-A*B*M**
12)*Q**2-A*B*N**2*P**2+A*B*M**2*N**2)
 GVO =
1(4*A26*M*P*Q**2+4*A26*M*N**2*P)/((P**2-M**2)*Q**2-N**2*P**2+
1M**2*N**2)
 AW =
1(((64*PI**2*A**2*H26-48*PI**2*A**2*F26*H**2)*N**3+((192*PI**
12*B**2*H16-144*PI**2*B**2*F16*H**2)*M**2+36*A**2*A45*B**2*H*
1*4-288*A**2*B**2*D45*H**2+576*A**2*B**2*F45)*N)*P*Q*R**2+((6
14*PI**2*A**2*J26-48*PI**2*A**2*H**2*H26)*N**3+(36*A**2*B**2*
1D26*H**4-48*A**2*B**2*F26*H**2)*N)*P*Q)/(((9*PI*A*B**2*H**4*
1P**2-9*PI*A*B**2*H**4*M**2)*Q**2-9*PI*A*B**2*H**4*N**2*P**2+
19*PI*A*B**2*H**4*M**2*N**2)*R**2)
 BW =
1(((192*PI**2*A**2*H26-144*PI**2*A**2*F26*H**2)*M*N**2+(64*PI
1**2*B**2*H16-48*PI**2*B**2*F16*H**2)*M**3+(36*A**2*A45*B**2*
1H**4-288*A**2*B**2*D45*H**2+576*A**2*B**2*F45)*M)*P*Q*R**2+(
164*PI**2*A**2*J26-48*PI**2*A**2*H**2*H26)*M*N**2*P*Q)/(((9*P
1I*A**2*B*H**4*P**2-9*PI*A**2*B*H**4*M**2)*Q**2-9*PI*A**2*B*H
1**4*N**2*P**2+9*PI*A**2*B*H**4*M**2*N**2)*R**2)
 CW =
1((256*PI**2*A**2*H26*M*N**3+(256*PI**2*B**2*H16*M**3+(72*A**
12*A45*B**2*H**4-576*A**2*B**2*D45*H**2+1152*A**2*B**2*F45)*M
1)*N)*P*Q*R**2+(128*PI**2*A**2*J26*M*N**3-96*A**2*B**2*F26*H*
1*2*M*N)*P*Q)/(((9*A**2*B**2*H**4*P**2-9*A**2*B**2*H**4*M**2)
1*Q**2-9*A**2*B**2*H**4*N**2*P**2+9*A**2*B**2*H**4*M**2*N**2)
1*R**2)
 EW =
1(8*PI**2*A**2*F26*N**3+(8*PI**2*B**2*F16*M**2-4*A**2*A26*B**
12*H**2)*N)*P*Q/(((PI*A*B**2*H**2*P**2-PI*A*B**2*H**2*M**2)*Q
1**2-PI*A*B**2*H**2*N**2*P**2+PI*A*B**2*H**2*M**2*N**2)*R)
 GW =
1(24*PI**2*A**2*F26*M*N**2-8*PI**2*B**2*F16*M**3-12*A**2*A26*
1B**2*H**2*M)*P*Q/(((3*PI*A**2*B*H**2*P**2-3*PI*A**2*B*H**2*M
1**2)*Q**2-3*PI*A**2*B*H**2*N**2*P**2+3*PI*A**2*B*H**2*M**2*N
1**2)*R)
 AJX =
1((128*H16+36*D16*H**4-144*F16*H**2)*N*P**2+(36*D16*H**4-48*F
116*H**2)*M**2*N)*Q/((9*H**4*P**2-9*H**4*M**2)*Q**2-9*H**4*N*
1*2*P**2+9*H**4*M**2*N**2)
 BJX =
1(((128*PI**2*B**2*H16+36*PI**2*B**2*D16*H**4-144*PI**2*B**2*
1F16*H**2)*M*P**2+(64*PI**2*A**2*H26+36*PI**2*A**2*D26*H**4-9
16*PI**2*A**2*F26*H**2)*M*N**2+(48*PI**2*B**2*F16*H**2-64*PI*
1*2*B**2*H16)*M**3+(36*A**2*A45*B**2*H**4-288*A**2*B**2*D45*H
1**2+576*A**2*B**2*F45)*M)*Q*R**2+(64*PI**2*A**2*J26-96*PI**2
1*A**2*H**2*H26+36*PI**2*A**2*F26*H**4)*M*N**2*Q)/(((9*PI**2*
1A*B*H**4*P**2-9*PI**2*A*B*H**4*M**2)*Q**2-9*PI**2*A*B*H**4*N
```

```
     1**2*P**2+9*PI**2*A*B*H**4*M**2*N**2)*R**2)
      CJX =
     1(((256*PI**2*B**2*H16-96*PI**2*B**2*F16*H**2)*M*N*P**2+(64*P
     1I**2*A**2*H26-48*PI**2*A**2*F26*H**2)*M*N**3+((-64*PI**2*B**
     12*H16-48*PI**2*B**2*F16*H**2)*M**3+(36*A**2*A45*B**2*H**4-28
     18*A**2*B**2*D45*H**2+576*A**2*B**2*F45)*M)*N)*Q*R**2+((64*PI
     1**2*A**2*J26-48*PI**2*A**2*H**2*H26)*M*N**3+(36*A**2*B**2*D2
     16*H**4-48*A**2*B**2*F26*H**2)*M*N)*Q)/(((9*PI*A*B**2*H**4*P*
     1*2-9*PI*A*B**2*H**4*M**2)*Q**2-9*PI*A*B**2*H**4*N**2*P**2+9*
     1PI*A*B**2*H**4*M**2*N**2)*R**2)
      EJX =
     1-((6*D16*H**2-16*F16)*N*P**2+(12*D16*H**2-8*F16)*M**2*N)*Q/(
     1((3*H**2*P**2-3*H**2*M**2)*Q**2-3*H**2*N**2*P**2+3*H**2*M**2
     1*N**2)*R)
      GJX =
     1((6*B**2*D16*H**2-16*B**2*F16)*M*P**2+(16*A**2*F26-12*A**2*D
     126*H**2)*M*N**2+8*B**2*F16*M**3)*Q/(((3*A*B*H**2*P**2-3*A*B*
     1H**2*M**2)*Q**2-3*A*B*H**2*N**2*P**2+3*A*B*H**2*M**2*N**2)*R
     1)
      AJY =
     1(((128*PI**2*A**2*H26+36*PI**2*A**2*D26*H**4-144*PI**2*A**2*
     1F26*H**2)*N*P*Q**2+((48*PI**2*A**2*F26*H**2-64*PI**2*A**2*H2
     16)*N**3+((64*PI**2*B**2*H16+36*PI**2*B**2*D16*H**4-96*PI**2*
     1B**2*F16*H**2)*M**2+36*A**2*A45*B**2*H**4-288*A**2*B**2*D45*
     1H**2+576*A**2*B**2*F45)*N)*P)*R**2+(128*PI**2*A**2*J26-144*P
     1I**2*A**2*H**2*H26+36*PI**2*A**2*F26*H**4)*N*P*Q**2+(48*PI**
     12*A**2*H**2*H26-64*PI**2*A**2*J26)*N**3*P)/(((9*PI**2*A*B*H*
     1*4*P**2-9*PI**2*A*B*H**4*M**2)*Q**2-9*PI**2*A*B*H**4*N**2*P*
     1*2+9*PI**2*A*B*H**4*M**2*N**2)*R**2)
      BJY =
     1((128*H26+36*D26*H**4-144*F26*H**2)*M*P*Q**2+(36*D26*H**4-48
     1*F26*H**2)*M*N**2*P)/((9*H**4*P**2-9*H**4*M**2)*Q**2-9*H**4*
     1N**2*P**2+9*H**4*M**2*N**2)
      CJY =
     1(((256*PI**2*A**2*H26-96*PI**2*A**2*F26*H**2)*M*N*P*Q**2+((-
     164*PI**2*A**2*H26-48*PI**2*A**2*F26*H**2)*M*N**3+((64*PI**2*
     1B**2*H16-48*PI**2*B**2*F16*H**2)*M**3+(36*A**2*A45*B**2*H**4
     1-288*A**2*B**2*D45*H**2+576*A**2*B**2*F45)*M)*N)*P)*R**2+(12
     18*PI**2*A**2*J26-48*PI**2*A**2*H**2*H26)*M*N*P*Q**2-64*PI**2
     1*A**2*J26*M*N**3*P)/(((9*PI*A**2*B*H**4*P**2-9*PI*A**2*B*H**
     14*M**2)*Q**2-9*PI*A**2*B*H**4*N**2*P**2+9*PI*A**2*B*H**4*M**
     12*N**2)*R**2)
      EJY =
     1-((6*A*D26*H**2-16*A*F26)*N*P*Q**2+8*A*F26*N**3*P)/(((B*H**2
     1*P**2-B*H**2*M**2)*Q**2-B*H**2*N**2*P**2+B*H**2*M**2*N**2)*R
     1)
      GJY =
     1-((6*D26*H**2-16*F26)*M*P*Q**2+8*F26*M*N**2*P)/(((3*H**2*P**
     12-3*H**2*M**2)*Q**2-3*H**2*N**2*P**2+3*H**2*M**2*N**2)*R)
C
      ELSE
C
```

```
      AUO = 0.0
      BUO = 0.0
      CUO = 0.0
      EUO = 0.0
      GUO = 0.0
      AVO = 0.0
      BVO = 0.0
      CVO = 0.0
      EVO = 0.0
      GVO = 0.0
      AW = 0.0
      BW = 0.0
      CW = 0.0
      EW = 0.0
      GW = 0.0
      AJX = 0.0
      BJX = 0.0
      CJX = 0.0
      EJX = 0.0
      GJX = 0.0
      AJY = 0.0
      BJY = 0.0
      CJY = 0.0
      EJY = 0.0
      GJY = 0.0
      ENDIF
C------------------------------------------------------------
C     STORE THESE TERMS IN THE STIFFNESS MATRIX
C------------------------------------------------------------
      STIFF(I,J) = AUO
      STIFF(I,J + MMAX * NMAX) = BUO
      STIFF(I,J + 2 * MMAX * NMAX) = CUO
      STIFF(I,J + 3 * MMAX * NMAX) = EUO
      STIFF(I,J + 4 * MMAX * NMAX) = GUO
      STIFF(I + MMAX * NMAX,J) = AVO
      STIFF(I + MMAX * NMAX,J + MMAX * NMAX) = BVO
      STIFF(I + MMAX * NMAX,J + 2 * MMAX * NMAX) = CVO
      STIFF(I + MMAX * NMAX,J + 3 * MMAX * NMAX) = EVO
      STIFF(I + MMAX * NMAX,J + 4 * MMAX * NMAX) = GVO
      STIFF(I + 2 * MMAX * NMAX,J) = AW
      STIFF(I + 2 * MMAX * NMAX,J + MMAX * NMAX) = BW
      STIFF(I + 2 * MMAX * NMAX,J + 2 * MMAX * NMAX) = CW
      STIFF(I + 2 * MMAX * NMAX,J + 3 * MMAX * NMAX) = EW
      STIFF(I + 2 * MMAX * NMAX,J + 4 * MMAX * NMAX) = GW
      STIFF(I + 3 * MMAX * NMAX,J) = AJX
      STIFF(I + 3 * MMAX * NMAX,J + MMAX * NMAX) = BJX
      STIFF(I + 3 * MMAX * NMAX,J + 2 * MMAX * NMAX) = CJX
      STIFF(I + 3 * MMAX * NMAX,J + 3 * MMAX * NMAX) = EJX
      STIFF(I + 3 * MMAX * NMAX,J + 4 * MMAX * NMAX) = GJX
      STIFF(I + 4 * MMAX * NMAX,J) = AJY
      STIFF(I + 4 * MMAX * NMAX,J + MMAX * NMAX) = BJY
      STIFF(I + 4 * MMAX * NMAX,J + 2 * MMAX * NMAX) = CJY
```

```
      STIFF(I + 4 * MMAX * NMAX,J + 3 * MMAX * NMAX) = EJY
      STIFF(I + 4 * MMAX * NMAX,J + 4 * MMAX * NMAX) = GJY
C------------------------------------------------------------
C     COMPUTE MASS MATRIX ELEMENTS
C------------------------------------------------------------
C     FIRST CALCULATE THE MASS MOMENTS OF INERTIA.
      I2BARPR = RHO * H**3 / (15.0 * R)
      I3BARPR = RHO * H**3 / (60.0 * R)
      I5BAR = RHO * H**3 * 4.0 / 315.0
      I7 = RHO * H**7 / 448.0
      I1 = RHO * H
      I4BAR = RHO * H**3 * 17.0 / 315.0
      AUOMASS = 0.0
      BUOMASS = 0.0
      CUOMASS = 0.0
      EUOMASS = 0.0
      GUOMASS = 0.0
      AVOMASS = 0.0
      EVOMASS = 0.0
      GVOMASS = 0.0
      EWMASS = 0.0
      GWMASS = 0.0
      BJXMASS = 0.0
      EJXMASS = 0.0
      GJXMASS = 0.0
      AJYMASS = 0.0
      EJYMASS = 0.0
      GJYMASS = 0.0
C
      IF(M .EQ. P .AND. N .EQ. Q) THEN
C
      IF(NBUCVIB .EQ. 1) THEN
C     VIBRATIONS PROBLEM - WE ARE LOOKING FOR THE NATURAL
C     FREQUENCIES
      BVOMASS = -1.0 * (
     1A*B*I2BARPR/4.0  )
      CVOMASS = -1.0 * (
     1-PI*A*I3BARPR*Q/4.0  )
      AWMASS = -1.0 *  (
     1-PI*B*I5BAR*P/4.0  )
      BWMASS = -1.0 *  (
     1-PI*A*I5BAR*Q/4.0  )
      CWMASS = -1.0 *  (
     1(16*PI**2*A**2*I7*Q**2+16*PI**2*B**2*I7*P**2+9*A**2*B**2*H**
     14*I1)/(A*B*H**4)/36.0  )
      AJXMASS = -1.0 * (
     1A*B*I4BAR/4.0  )
      CJXMASS = -1.0 * (
     1-PI*B*I5BAR*P/4.0  )
      BJYMASS = -1.0 * (
     1A*B*I4BAR/4.0  )
      CJYMASS = -1.0 * (
```

```
     1-PI*A*I5BAR*Q/4.0  )
      ELSE
C     BUCKLING PROBLEM - WE ARE LOOKING FOR THE CRITICAL BUCKLING
C     LOADS
      BVOMASS = 0.0
      CVOMASS = 0.0
      AWMASS = 0.0
      BWMASS = 0.0
      CWMASS = -1.0 * (
     1-PI**2*B*P**2/A/4.0  )
      AJXMASS = 0.0
      CJXMASS = 0.0
      BJYMASS = 0.0
      CJYMASS = 0.0
      ENDIF
C
      ELSE
C
      BVOMASS = 0.0
      CVOMASS = 0.0
      AWMASS = 0.0
      BWMASS = 0.0
      CWMASS = 0.0
      AJXMASS = 0.0
      CJXMASS = 0.0
      BJYMASS = 0.0
      CJYMASS = 0.0
C
      ENDIF
C-----------------------------------------------------------------
C     STORE THESE TERMS IN THE MASS MATRIX
C-----------------------------------------------------------------
      MASS(I,J) = AUOMASS
      MASS(I,J + MMAX * NMAX) = BUOMASS
      MASS(I,J + 2 * MMAX * NMAX) = CUOMASS
      MASS(I,J + 3 * MMAX * NMAX) = EUOMASS
      MASS(I,J + 4 * MMAX * NMAX) = GUOMASS
      MASS(I + MMAX * NMAX,J) = AVOMASS
      MASS(I + MMAX * NMAX,J + MMAX * NMAX) = BVOMASS
      MASS(I + MMAX * NMAX,J + 2 * MMAX * NMAX) = CVOMASS
      MASS(I + MMAX * NMAX,J + 3 * MMAX * NMAX) = EVOMASS
      MASS(I + MMAX * NMAX,J + 4 * MMAX * NMAX) = GVOMASS
      MASS(I + 2 * MMAX * NMAX,J) = AWMASS
      MASS(I + 2 * MMAX * NMAX,J + MMAX * NMAX) = BWMASS
      MASS(I + 2 * MMAX * NMAX,J + 2 * MMAX * NMAX) = CWMASS
      MASS(I + 2 * MMAX * NMAX,J + 3 * MMAX * NMAX) = EWMASS
      MASS(I + 2 * MMAX * NMAX,J + 4 * MMAX * NMAX) = GWMASS
      MASS(I + 3 * MMAX * NMAX,J) = AJXMASS
      MASS(I + 3 * MMAX * NMAX,J + MMAX * NMAX) = BJXMASS
      MASS(I + 3 * MMAX * NMAX,J + 2 * MMAX * NMAX) = CJXMASS
      MASS(I + 3 * MMAX * NMAX,J + 3 * MMAX * NMAX) = EJXMASS
      MASS(I + 3 * MMAX * NMAX,J + 4 * MMAX * NMAX) = GJXMASS
```

```
      MASS(I + 4 * MMAX * NMAX,J) = AJYMASS
      MASS(I + 4 * MMAX * NMAX,J + MMAX * NMAX) = BJYMASS
      MASS(I + 4 * MMAX * NMAX,J + 2 * MMAX * NMAX) = CJYMASS
      MASS(I + 4 * MMAX * NMAX,J + 3 * MMAX * NMAX) = EJYMASS
      MASS(I + 4 * MMAX * NMAX,J + 4 * MMAX * NMAX) = GJYMASS
C-----------------------------------------------------------------
      J = J + 1
   20 CONTINUE
      I = I + 1
      J = 1
   10 CONTINUE
C-----------------------------------------------------------------
C     CALL THE IMSL LIBRARY SUBROUTINE. USE THE MASS AND STIFFNESS
C     MATRICES AS INPUT AND FIND THE EIGENVALUES AND EIGENVECTORS.
      CALL DGVCRG(MSIZE,STIFF,MSIZE,MASS,MSIZE,ALPHA,BETA,EVEC,
     1MSIZE)
      DO 40 I = 1,MSIZE
      IF(BETA(I) .NE. 0.0) THEN
      EVAL(I) = ALPHA(I) / BETA(I)
      ELSE
      EVAL(I) = (1.0D+30 , 0.0D+00)
      ENDIF
   40 CONTINUE
      IF(NBUCVIB .EQ. 1) THEN
C
C     PRINT OUT THE FIRST 10 MODES FOR THE VIBRATION PROBLEM
C
      DO 50 I = 1,10
      REVAL = DREAL(EVAL(I))
      AGEVAL = DIMAG(EVAL(I))
      IF(ABS(AGEVAL) .GT. 1.0D-15) THEN
      WRITE(2,115) I
      ELSEIF(REVAL .GT. 1.0D+28) THEN
      WRITE(2,125) I
      ELSEIF(REVAL .LT. 0.0) THEN
      WRITE(2,120) I
      ELSE
      OMEGA = SQRT(REVAL)
      WRITE(2,130) I,REVAL,OMEGA
      ENDIF
   50 CONTINUE
C
      ELSE
C
C     PRINT OUT THE CRITICAL BUCKLING LOAD. THE CRITICAL
C     BUCKLING LOAD IS THE EIGENVALUE WITH THE SMALLEST ABSOLUTE
C     VALUE.
C
      DO 55 I = 2,MSIZE
      IF(ABS(DIMAG(EVAL(I-1))) .GT. 1.0D-15) THEN
      GO TO 55
      ENDIF
```

```
      IF(ABS(DREAL(EVAL(I))) .GT. ABS(DREAL(EVAL(I-1))) .AND. ABS(
     1DREAL(EVAL(I-1))) .LT. 1.0D+28) THEN
      WRITE(2,220) DREAL(EVAL(I-1))
      ENDIF
   55 CONTINUE
C
      ENDIF
C
C     PRINT OUT THE 1ST MODE OF THE DEFLECTION, W(X,Y), ALONG THE
C     MIDLINES OF THE PANEL:  X = A/2 AND Y = B/2
C
C     PRINT OUT THE W EIGENVECTOR, CMN
C
      II = 1
      WRITE(2,500)
      WRITE(2,510)
      MNWMIN = 1 + 2 * MMAX * NMAX
      MNWMAX = 3 * MMAX * NMAX
      DO 400 I = MNWMIN, MNWMAX
      REVEC(II) = DREAL(EVEC(I,1))
      AGEVEC = DIMAG(EVEC(I,1))
      IF(ABS(AGEVEC) .GT. 1.0D-15) THEN
      WRITE(2,520) I,II,REVEC(II)
      ELSE
      WRITE(2,530) I,II,REVEC(II)
      ENDIF
      II = II + 1
  400 CONTINUE
C
C     DETERMINE W(X=A/2,Y)
C
      ASTEP = A / 50.0
      BSTEP = B / 50.0
      XCOORD = A / 2.0
      YCOORD = 0.0
      WRITE(2,540)
      WRITE(2,542)
  801 WMODE = 0.0
      JJJ = 1
      DO 470 M = 1,MMAX
      DO 472 N = 1,NMAX
      WMODE = WMODE + REVEC(JJJ)*SIN(M*PI*XCOORD/A)*SIN(N*PI*YCOOR
     1D/B)
      JJJ = JJJ + 1
  472 CONTINUE
  470 CONTINUE
      WRITE(2,550) YCOORD,WMODE
      YCOORD = YCOORD + BSTEP
      IF(YCOORD .GT. B) THEN
      GO TO 800
      ELSE
      GO TO 801
```

```
      ENDIF
C
800   YCOORD = B / 2.0
C
C     DETERMINE W(X,Y=B/2)
C
      XCOORD = 0.0
      WRITE(2,560)
      WRITE(2,570)
810   WMODE = 0.0
      JJJ = 1
      DO 480 M = 1, MMAX
      DO 482 N = 1, NMAX
      WMODE = WMODE + REVEC(JJJ)*SIN(M*PI*XCOORD/A)*SIN(N*PI*YCOOR
     1D/B)
      JJJ = JJJ + 1
482   CONTINUE
480   CONTINUE
      WRITE(2,550)XCOORD,WMODE
      XCOORD = XCOORD + ASTEP
      IF(XCOORD .GT. A) THEN
      GO TO 850
      ELSE
      GO TO 810
      ENDIF
C----------------------------------------------------------------
115   FORMAT(/,9X,I3,11X,'EIGENVALUE IS COMPLEX')
120   FORMAT(/,9X,I3,11X,'EIGENVALUE IS NEGATIVE')
125   FORMAT(/,9X,I3,11X,'EIGENVALUE IS INFINITE')
130   FORMAT(/,9X,I3,10X,D20.13,12X,D20.13)
200   FORMAT(/,9X,I3,10X,D20.13)
220   FORMAT(//,5X,'CRITICAL BUCKLING LOAD = ',1X,D20.13)
500   FORMAT(//,5X,'W EIGENVECTOR, CMN, FOR 1ST MODE')
510   FORMAT(//,5X,'M,N',10X,'CMN')
520   FORMAT(/,5X,I4,2X,I4,12X,D20.13,3X,'COMPLEX')
530   FORMAT(/,5X,I4,2X,I4,12X,D20.13)
540   FORMAT(//,5X,'DEFLECTION, W(X=A/2,Y)')
542   FORMAT(//,5X,'Y (IN.)',10X,'W(A/2,Y) (IN.)')
550   FORMAT(/,5X,F6.2,11X,E15.8)
560   FORMAT(//,5X,'DEFLECTION, W(X,Y=B/2)')
570   FORMAT(//,5X,'X (IN.)',10X,'W(X,B/2) (IN.)')
850   RETURN
      END
```

The next listing is the code for clamped boundary conditions. As stated in chapter III, the only part of the program that is boundary condition dependant is subroutine "GALERK" (with the exception of a few format statements in the main program).

****** Clamped Boundary Condition ******

```
C-------------------------------------------------------------------
      SUBROUTINE GALERK(PI,R,H,A,B,A11,A12,A22,A16,A26,A66,A44,A45
     1,A55,D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26
     2,F66,F44,F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26
     3,J66,NBUCVIB,MMAX,MSIZE,RHO,STIFF,MASS,BETA,ALPHA,EVAL,EVEC,
     4MSIZESQ,REVEC)
C-------------------------------------------------------------------
C     THIS SUBROUTINE GENERATES THE GALERKIN EQUATIONS AND FORMS
C     THE MASS AND STIFFNESS MATRICES.  THEN IT CALLS DGVCRG, AN
C     IMSL SUBROUTINE WHICH SOLVES THE EIGENVALUE PROBLEM:
C              [STIFF]{X} = (OMEGA^2 OR N1BAR)[MASS]{X}.
C-------------------------------------------------------------------
      DOUBLE PRECISION  PI,R,H,A,B,A11,A12,A22,A16,A26,A66,A44,
     1A45,A55,D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,
     2F26,F66,F44,F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,
     3J26,J66,STIFF(MSIZE,MSIZE),MASS(MSIZE,MSIZE),AUO,BUO,CUO,EUO
     4,GUO,AVO,BVO,CVO,EVO,GVO,AW,BW,CW,EW,GW,AJX,BJX,CJX,EJX,GJX,
     5AJY,BJY,CJY,EJY,GJY,AUOMASS,BUOMASS,CUOMASS,EUOMASS,GUOMASS,
     6AVOMASS,BVOMASS,CVOMASS,EVOMASS,GVOMASS,AWMASS,BWMASS,CWMASS
     7,EWMASS,GWMASS,AJXMASS,BJXMASS,CJXMASS,EJXMASS,GJXMASS,AJYMA
     8SS,BJYMASS,CJYMASS,EJYMASS,GJYMASS,RHO,I2BARPR,I3BARPR,I5BAR
     9,I7,I1,I4BAR
      INTEGER P,Q
C     THESE VARIABLES NEEDED FOR THE IMSL EIGENVALUE SOLVER.
      DOUBLE PRECISION BETA(MSIZE),REVAL,OMEGA,AGEVAL,AGEVEC,
     1REVEC(MSIZESQ)
      DOUBLE COMPLEX ALPHA(MSIZE),EVAL(MSIZE),EVEC(MSIZE,MSIZE)
C-------------------------------------------------------------------
C     NUMBER OF TERMS IN THE ADMISSIBLE FUNCTIONS
      NMAX = MMAX
C     GENERATE GALERKIN EQUATIONS
      I = 1
      J = 1
      DO 10 P = 1,MMAX
      DO 10 Q = 1,NMAX
      DO 20 M = 1,MMAX
      DO 20 N = 1,NMAX
C-------------------------------------------------------------------
C     COMPUTE STIFFNESS MATRIX ELEMENTS
C-------------------------------------------------------------------
      IF (M .EQ. P .AND. N .EQ. Q) THEN
C
      AUO = 0.0
      BUO = 0.0
      CUO =
     1-((8*F66+4*F12)*P*PI**3*Q**2-3*A12*B**2*H**2*P*PI)/(B*H**2*R
     1)/12.0
      EUO =
```

```
1-((4*A**2*A66*PI**2*Q**2+4*A11*B**2*P**2*PI**2)*R**2+A**2*D6
16*PI**2*Q**2)/(A*B*R**2)/16.0
 GUO =
1-((4*A66+4*A12)*P*PI**2*Q*R**2-D66*P*PI**2*Q)/R**2/16.0
 AVO = 0.0
 BVO = 0.0
 CVO =
1-(4*A*F22*PI**3*Q**3-3*A*A22*B**2*H**2*PI*Q)/(B**2*H**2*R)/1
12.0
 EVO =
1-((4*A66+4*A12)*P*PI**2*Q*R**2-D66*P*PI**2*Q)/R**2/16.0
 GVO =
1-((4*A**2*A22*PI**2*Q**2+4*A66*B**2*P**2*PI**2)*R**2+B**2*D6
16*P**2*PI**2)/(A*B*R**2)/16.0
 AW = 0.0
 BW = 0.0
 CW =
1-((16*A**4*H22*PI**4*Q**4+((64*A**2*B**2*H66+32*A**2*B**2*H1
12)*P**2*PI**4+(9*A**4*A44*B**2*H**4-72*A**4*B**2*D44*H**2+14
14*A**4*B**2*F44)*PI**2)*Q**2+16*B**4*H11*P**4*PI**4+(9*A**2*
1A55*B**4*H**4-72*A**2*B**4*D55*H**2+144*A**2*B**4*F55)*P**2*
1PI**2)*R**2+16*A**4*J22*PI**4*Q**4+(16*A**2*B**2*J66*P**2*PI
1**4-24*A**4*B**2*F22*H**2*PI**2)*Q**2+9*A**4*A22*B**4*H**4)/
1(A**3*B**3*H**4*R**2)/36.0
 EW =
1-((8*F66+4*F12)*P*PI**3*Q**2-3*A12*B**2*H**2*P*PI)/(B*H**2*R
1)/12.0
 GW =
1-(4*A*F22*PI**3*Q**3-3*A*A22*B**2*H**2*PI*Q)/(B**2*H**2*R)/1
12.0
 AJX =
1-(((16*A**2*H66+9*A**2*D66*H**4-24*A**2*F66*H**2)*PI**2*Q**2
1+(16*B**2*H11+9*B**2*D11*H**4-24*B**2*F11*H**2)*P**2*PI**2+9
1*A**2*A55*B**2*H**4-72*A**2*B**2*D55*H**2+144*A**2*B**2*F55)
1*R**2+(16*A**2*J66-24*A**2*H**2*H66+9*A**2*F66*H**4)*PI**2*Q
1**2)/(A*B*H**4*R**2)/36.0
 BJX =
1-(((16*A**2*H26+9*A**2*D26*H**4-24*A**2*F26*H**2)*PI**2*Q**2
1+(16*B**2*H16+9*B**2*D16*H**4-24*B**2*F16*H**2)*P**2*PI**2+9
1*A**2*A45*B**2*H**4-72*A**2*B**2*D45*H**2+144*A**2*B**2*F45)
1*R**2+(16*A**2*J26-24*A**2*H**2*H26+9*A**2*F26*H**4)*PI**2*Q
1**2)/(A*B*H**4*R**2)/36.0
 CJX = 0.0
 EJX = 0.0
 GJX = 0.0
 AJY =
1-(((16*A**2*H26+9*A**2*D26*H**4-24*A**2*F26*H**2)*PI**2*Q**2
1+(16*B**2*H16+9*B**2*D16*H**4-24*B**2*F16*H**2)*P**2*PI**2+9
1*A**2*A45*B**2*H**4-72*A**2*B**2*D45*H**2+144*A**2*B**2*F45)
1*R**2+(16*A**2*J26-24*A**2*H**2*H26+9*A**2*F26*H**4)*PI**2*Q
1**2)/(A*B*H**4*R**2)/36.0
 BJY =
```

```
     1-(((16*A**2*H22+9*A**2*D22*H**4-24*A**2*F22*H**2)*PI**2*Q**2
     1+(16*B**2*H66+9*B**2*D66*H**4-24*B**2*F66*H**2)*P**2*PI**2+9
     1*A**2*A44*B**2*H**4-72*A**2*B**2*D44*H**2+144*A**2*B**2*F44)
     1*R**2+(16*A**2*J22-24*A**2*H**2*H22+9*A**2*F22*H**4)*PI**2*Q
     1**2)/(A*B*H**4*R**2)/36.0
      CJY = 0.0
      EJY = 0.0
      GJY = 0.0
C
      ELSEIF (M .EQ. P .AND. MOD(N + Q,2) .NE. 0) THEN
C
      AUO =
     1-(3*D16*H**2-4*F16)*N*P*PI*Q/((2*H**2*Q**2-2*H**2*N**2)*R)
      BUO =
     1-((3*D66+6*D12)*H**2-4*F66-8*F12)*N*P*PI*Q/((6*H**2*Q**2-6*H
     1**2*N**2)*R)
      CUO = 0.0
      EUO = 0.0
      GUO = 0.0
      AVO =
     1-((6*A**2*D26*H**2-8*A**2*F26)*N*PI*Q**2+(4*B**2*F16-3*B**2*
     1D16*H**2)*N*P**2*PI)/((6*A*B*H**2*Q**2-6*A*B*H**2*N**2)*R)
      BVO =
     1-((6*A**2*D22*H**2-8*A**2*F22)*N*PI*Q**2+(4*B**2*F66-3*B**2*
     1D66*H**2)*N*P**2*PI)/((6*A*B*H**2*Q**2-6*A*B*H**2*N**2)*R)
      CVO = 0.0
      EVO = 0.0
      GVO = 0.0
      AW =
     1((((48*B**2*H16-36*B**2*F16*H**2)*N*P**2+(16*A**2*H26-12*A**
     12*F26*H**2)*N**3)*PI**2+(9*A**2*A45*B**2*H**4-72*A**2*B**2*D
     145*H**2+144*A**2*B**2*F45)*N)*Q*R**2+((16*A**2*J26-12*A**2*H
     1**2*H26)*N**3*PI**2+(9*A**2*B**2*D26*H**4-12*A**2*B**2*F26*H
     1**2)*N)*Q)/((9*A*B**2*H**4*Q**2-9*A*B**2*H**4*N**2)*R**2)
      BW =
     1((((32*B**2*H66+16*B**2*H12+(-24*B**2*F66-12*B**2*F12)*H**2)
     1*N*P**2+(16*A**2*H22-12*A**2*F22*H**2)*N**3)*PI**2+(9*A**2*A
     144*B**2*H**4-72*A**2*B**2*D44*H**2+144*A**2*B**2*F44)*N)*Q*R
     1**2+((16*A**2*J22-12*A**2*H**2*H22)*N**3*PI**2+(9*A**2*B**2*
     1D22*H**4-12*A**2*B**2*F22*H**2)*N)*Q)/((9*A*B**2*H**4*Q**2-9
     1*A*B**2*H**4*N**2)*R**2)
      CW = 0.0
      EW = 0.0
      GW = 0.0
      AJX = 0.0
      BJX = 0.0
      CJX =
     1-((((48*B**2*H16-36*B**2*F16*H**2)*N*P**2+(16*A**2*H26-12*A*
     1*2*F26*H**2)*N**3)*PI**2+(9*A**2*A45*B**2*H**4-72*A**2*B**2*
     1D45*H**2+144*A**2*B**2*F45)*N)*Q*R**2+((16*A**2*J26-12*A**2*
     1H**2*H26)*N**3*PI**2+(9*A**2*B**2*D26*H**4-12*A**2*B**2*F26*
     1H**2)*N)*Q)/((9*A*B**2*H**4*Q**2-9*A*B**2*H**4*N**2)*R**2)
```

```
      EJX =
     1(3*D16*H**2-4*F16)*N*P*PI*Q/((2*H**2*Q**2-2*H**2*N**2)*R)
      GJX =
     1-((3*B**2*D16*H**2-4*B**2*F16)*P**2+(8*A**2*F26-6*A**2*D26*H
     1**2)*N**2)*PI*Q/((6*A*B*H**2*Q**2-6*A*B*H**2*N**2)*R)
      AJY = 0.0
      BJY = 0.0
      CJY =
     1-(((((32*B**2*H66+16*B**2*H12+(-24*B**2*F66-12*B**2*F12)*H**2
     1)*N*P**2+(16*A**2*H22-12*A**2*F22*H**2)*N**3)*PI**2+(9*A**2*
     1A44*B**2*H**4-72*A**2*B**2*D44*H**2+144*A**2*B**2*F44)*N)*Q*
     1R**2+((16*A**2*J22-12*A**2*H**2*H22)*N**3*PI**2+(9*A**2*B**2
     1*D22*H**4-12*A**2*B**2*F22*H**2)*N)*Q)/((9*A*B**2*H**4*Q**2-
     19*A*B**2*H**4*N**2)*R**2)
      EJY =
     1((3*D66+6*D12)*H**2-4*F66-8*F12)*N*P*PI*Q/((6*H**2*Q**2-6*H*
     1*2*N**2)*R)
      GJY =
     1-((3*B**2*D66*H**2-4*B**2*F66)*P**2+(8*A**2*F22-6*A**2*D22*H
     1**2)*N**2)*PI*Q/((6*A*B*H**2*Q**2-6*A*B*H**2*N**2)*R)
C
      ELSEIF(MOD(M + P,2) .NE. 0 .AND. N .EQ. Q) THEN
C
      AUO =
     1-(3*A*D66*H**2-4*A*F66)*M*PI*Q**2/((2*B*H**2*P**2-2*B*H**2*M
     1**2)*R)
      BUO =
     1-(3*A*D26*H**2-4*A*F26)*M*PI*Q**2/((2*B*H**2*P**2-2*B*H**2*M
     1**2)*R)
      CUO = 0.0
      EUO = 0.0
      GUO = 0.0
      AVO =
     1-(3*D66*H**2-4*F66)*M*P*PI*Q/((6*H**2*P**2-6*H**2*M**2)*R)
      BVO =
     1-(3*D26*H**2-4*F26)*M*P*PI*Q/((6*H**2*P**2-6*H**2*M**2)*R)
      CVO = 0.0
      EVO = 0.0
      GVO = 0.0
      AW =
     1(((32*A**2*H66+16*A**2*H12+(-24*A**2*F66-12*A**2*F12)*H**2)*
     1M*P*PI**2*Q**2+(16*B**2*H11-12*B**2*F11*H**2)*M**3*P*PI**2+(
     19*A**2*A55*B**2*H**4-72*A**2*B**2*D55*H**2+144*A**2*B**2*F55
     1)*M*P)*R**2+(16*A**2*J66-12*A**2*H**2*H66)*M*P*PI**2*Q**2)/(
     1(9*A**2*B*H**4*P**2-9*A**2*B*H**4*M**2)*R**2)
      BW =
     1(((48*A**2*H26-36*A**2*F26*H**2)*M*P*PI**2*Q**2+(16*B**2*H16
     1-12*B**2*F16*H**2)*M**3*P*PI**2+(9*A**2*A45*B**2*H**4-72*A**
     12*B**2*D45*H**2+144*A**2*B**2*F45)*M*P)*R**2+(16*A**2*J26-12
     1*A**2*H**2*H26)*M*P*PI**2*Q**2)/((9*A**2*B*H**4*P**2-9*A**2*
     1B*H**4*M**2)*R**2)
      CW = 0.0
```

```
      EW = 0.0
      GW = 0.0
      AJX = 0.0
      BJX = 0.0
      CJX =
     1-(((32*A**2*H66+16*A**2*H12+(-24*A**2*F66-12*A**2*F12)*H**2)
     1*M*P*PI**2*Q**2+(16*B**2*H11-12*B**2*F11*H**2)*M**3*P*PI**2+
     1(9*A**2*A55*B**2*H**4-72*A**2*B**2*D55*H**2+144*A**2*B**2*F5
     15)*M*P)*R**2+(16*A**2*J66-12*A**2*H**2*H66)*M*P*PI**2*Q**2)/
     1((9*A**2*B*H**4*P**2-9*A**2*B*H**4*M**2)*R**2)
      EJX =
     1(3*A*D66*H**2-4*A*F66)*P*PI*Q**2/((2*B*H**2*P**2-2*B*H**2*M*
     1*2)*R)
      GJX =
     1(3*D66*H**2-4*F66)*M*P*PI*Q/((6*H**2*P**2-6*H**2*M**2)*R)
      AJY = 0.0
      BJY = 0.0
      CJY =
     1-(((48*A**2*H26-36*A**2*F26*H**2)*M*P*PI**2*Q**2+(16*B**2*H1
     16-12*B**2*F16*H**2)*M**3*P*PI**2+(9*A**2*A45*B**2*H**4-72*A*
     1*2*B**2*D45*H**2+144*A**2*B**2*F45)*M*P)*R**2+(16*A**2*J26-1
     12*A**2*H**2*H26)*M*P*PI**2*Q**2)/((9*A**2*B*H**4*P**2-9*A**2
     1*B*H**4*M**2)*R**2)
      EJY =
     1(3*A*D26*H**2-4*A*F26)*P*PI*Q**2/((2*B*H**2*P**2-2*B*H**2*M*
     1*2)*R)
      GJY =
     1(3*D26*H**2-4*F26)*M*P*PI*Q/((6*H**2*P**2-6*H**2*M**2)*R)
C
      ELSEIF(MOD(M + P,2) .NE. 0 .AND. MOD(N + Q,2) .NE. 0) THEN
C
      AUO = 0.0
      BUO = 0.0
      CUO =
     1((16*B**2*F16*M*N*P**2+24*A**2*F26*M*N**3+8*B**2*F16*M**3*N)
     1*PI**2-12*A**2*A26*B**2*H**2*M*N)*Q/((((3*A*B**2*H**2*P**2-3*
     1A*B**2*H**2*M**2)*PI*Q**2+(3*A*B**2*H**2*M**2*N**2-3*A*B**2*
     1H**2*N**2*P**2)*PI)*R)
      EUO =
     1(4*A16*N*P**2+4*A16*M**2*N)*Q/((P**2-M**2)*Q**2-N**2*P**2+M*
     1*2*N**2)
      GUO =
     1(4*A16*B**2*M*P**2+4*A**2*A26*M*N**2)*Q/((A*B*P**2-A*B*M**2)
     1*Q**2-A*B*N**2*P**2+A*B*M**2*N**2)
      AVO = 0.0
      BVO = 0.0
      CVO =
     1(16*A**2*F26*M*N*P*PI**2*Q**2+(8*A**2*F26*M*N**3-8*B**2*F16*
     1M**3*N)*P*PI**2-12*A**2*A26*B**2*H**2*M*N*P)/((((3*A**2*B*H**
     12*P**2-3*A**2*B*H**2*M**2)*PI*Q**2+(3*A**2*B*H**2*M**2*N**2-
     13*A**2*B*H**2*N**2*P**2)*PI)*R)
      EVO =
```

```
     1(4*A**2*A26*N*P*Q**2+4*A16*B**2*M**2*N*P)/((A*B*P**2-A*B*M**
     12)*Q**2-A*B*N**2*P**2+A*B*M**2*N**2)
      GVO =
     1(4*A26*M*P*Q**2+4*A26*M*N**2*P)/((P**2-M**2)*Q**2-N**2*P**2+
     1M**2*N**2)
      AW = 0.0
      BW = 0.0
      CW =
     1(((256*A**2*H26*M*N**3+256*B**2*H16*M**3*N)*P*PI**2+(72*A**2
     1*A45*B**2*H**4-576*A**2*B**2*D45*H**2+1152*A**2*B**2*F45)*M*
     1N*P)*Q*R**2+(128*A**2*J26*M*N**3*P*PI**2-96*A**2*B**2*F26*H*
     1*2*M*N*P)*Q)/(((9*A**2*B**2*H**4*P**2-9*A**2*B**2*H**4*M**2)
     1*Q**2-9*A**2*B**2*H**4*N**2*P**2+9*A**2*B**2*H**4*M**2*N**2)
     1*R**2)
      EW =
     1((8*A**2*F26*N**3+8*B**2*F16*M**2*N)*P*PI**2-4*A**2*A26*B**2
     1*H**2*N*P)*Q/(((A*B**2*H**2*P**2-A*B**2*H**2*M**2)*PI*Q**2+(
     1A*B**2*H**2*M**2*N**2-A*B**2*H**2*N**2*P**2)*PI)*R)
      GW =
     1((24*A**2*F26*M*N**2-8*B**2*F16*M**3)*P*PI**2-12*A**2*A26*B*
     1*2*H**2*M*P)*Q/(((3*A**2*B*H**2*P**2-3*A**2*B*H**2*M**2)*PI*
     1Q**2+(3*A**2*B*H**2*M**2*N**2-3*A**2*B*H**2*N**2*P**2)*PI)*R
     1)
      AJX =
     1(128*H16+72*D16*H**4-192*F16*H**2)*M*N*P*Q/((9*H**4*P**2-9*H
     1**4*M**2)*Q**2-9*H**4*N**2*P**2+9*H**4*M**2*N**2)
      BJX =
     1(64*H66+64*H12+(36*D66+36*D12)*H**4+(-96*F66-96*F12)*H**2)*M
     1*N*P*Q/((9*H**4*P**2-9*H**4*M**2)*Q**2-9*H**4*N**2*P**2+9*H*
     1*4*M**2*N**2)
      CJX = 0.0
      EJX = 0.0
      GJX = 0.0
      AJY =
     1(64*H66+64*H12+(36*D66+36*D12)*H**4+(-96*F66-96*F12)*H**2)*M
     1*N*P*Q/((9*H**4*P**2-9*H**4*M**2)*Q**2-9*H**4*N**2*P**2+9*H*
     1*4*M**2*N**2)
      BJY =
     1(128*H26+72*D26*H**4-192*F26*H**2)*M*N*P*Q/((9*H**4*P**2-9*H
     1**4*M**2)*Q**2-9*H**4*N**2*P**2+9*H**4*M**2*N**2)
      CJY = 0.0
      EJY = 0.0
      GJY = 0.0
C
      ELSE
C
      AUO = 0.0
      BUO = 0.0
      CUO = 0.0
      EUO = 0.0
      GUO = 0.0
      AVO = 0.0
```

```
      BVO = 0.0
      CVO = 0.0
      EVO = 0.0
      GVO = 0.0
      AW = 0.0
      BW = 0.0
      CW = 0.0
      EW = 0.0
      GW = 0.0
      AJX = 0.0
      BJX = 0.0
      CJX = 0.0
      EJX = 0.0
      GJX = 0.0
      AJY = 0.0
      BJY = 0.0
      CJY = 0.0
      EJY = 0.0
      GJY = 0.0
      ENDIF
C-----------------------------------------------------------------
C     STORE THESE TERMS IN THE STIFFNESS MATRIX
C-----------------------------------------------------------------
      STIFF(I,J) = AUO
      STIFF(I,J + MMAX * NMAX) = BUO
      STIFF(I,J + 2 * MMAX * NMAX) = CUO
      STIFF(I,J + 3 * MMAX * NMAX) = EUO
      STIFF(I,J + 4 * MMAX * NMAX) = GUO
      STIFF(I + MMAX * NMAX,J) = AVO
      STIFF(I + MMAX * NMAX,J + MMAX * NMAX) = BVO
      STIFF(I + MMAX * NMAX,J + 2 * MMAX * NMAX) = CVO
      STIFF(I + MMAX * NMAX,J + 3 * MMAX * NMAX) = EVO
      STIFF(I + MMAX * NMAX,J + 4 * MMAX * NMAX) = GVO
      STIFF(I + 2 * MMAX * NMAX,J) = AW
      STIFF(I + 2 * MMAX * NMAX,J + MMAX * NMAX) = BW
      STIFF(I + 2 * MMAX * NMAX,J + 2 * MMAX * NMAX) = CW
      STIFF(I + 2 * MMAX * NMAX,J + 3 * MMAX * NMAX) = EW
      STIFF(I + 2 * MMAX * NMAX,J + 4 * MMAX * NMAX) = GW
      STIFF(I + 3 * MMAX * NMAX,J) = AJX
      STIFF(I + 3 * MMAX * NMAX,J + MMAX * NMAX) = BJX
      STIFF(I + 3 * MMAX * NMAX,J + 2 * MMAX * NMAX) = CJX
      STIFF(I + 3 * MMAX * NMAX,J + 3 * MMAX * NMAX) = EJX
      STIFF(I + 3 * MMAX * NMAX,J + 4 * MMAX * NMAX) = GJX
      STIFF(I + 4 * MMAX * NMAX,J) = AJY
      STIFF(I + 4 * MMAX * NMAX,J + MMAX * NMAX) = BJY
      STIFF(I + 4 * MMAX * NMAX,J + 2 * MMAX * NMAX) = CJY
      STIFF(I + 4 * MMAX * NMAX,J + 3 * MMAX * NMAX) = EJY
      STIFF(I + 4 * MMAX * NMAX,J + 4 * MMAX * NMAX) = GJY
C-----------------------------------------------------------------
C     COMPUTE MASS MATRIX ELEMENTS
C-----------------------------------------------------------------
C     FIRST CALCULATE THE MASS MOMENTS OF INERTIA.
```

```
      I2BARPR = RHO * H**3 / (15.0 * R)
      I3BARPR = RHO * H**3 / (60.0 * R)
      I5BAR = RHO * H**3 * 4.0 / 315.0
      I7 = RHO * H**7 / 448.0
      I1 = RHO * H
      I4BAR = RHO * H**3 * 17.0 / 315.0
      AUOMASS = 0.0
      BUOMASS = 0.0
      CUOMASS = 0.0
      EUOMASS = 0.0
      GUOMASS = 0.0
      AVOMASS = 0.0
      EVOMASS = 0.0
      GVOMASS = 0.0
      EWMASS = 0.0
      GWMASS = 0.0
      BJXMASS = 0.0
      EJXMASS = 0.0
      GJXMASS = 0.0
      AJYMASS = 0.0
      EJYMASS = 0.0
      GJYMASS = 0.0
C
      IF(NBUCVIB .EQ. 1) THEN
C
C     VIBRATIONS PROBLEM - WE ARE LOOKING FOR THE NATURAL
C     FREQUENCIES
      IF(M .EQ. P .AND. N .EQ. Q) THEN
      CVOMASS = -1.0 * (
     1-A*I3BARPR*PI*Q/4.0 )
      CWMASS = -1.0 * (
     1(16*A**2*I7*PI**2*Q**2+16*B**2*I7*P**2*PI**2+9*A**2*B**2*H**
     14*I1)/(A*B*H**4)/36.0 )
      AJXMASS = -1.0 * (
     1A*B*I4BAR/4.0 )
      BJYMASS = -1.0 * (
     1A*B*I4BAR/4.0 )
      BVOMASS = 0.0
      BWMASS = 0.0
      CJYMASS = 0.0
      AWMASS = 0.0
      CJXMASS = 0.0
      ELSEIF(M .EQ. P .AND. MOD(N + Q,2) .NE. 0) THEN
      BVOMASS = -1.0 * (
     1-A*B*I2BARPR*N/(PI*Q**2-N**2*PI) )
      BWMASS = -1.0 * (
     1A*I5BAR*N*Q/(Q**2-N**2) )
      CJYMASS = -1.0 * (
     1-A*I5BAR*N*Q/(Q**2-N**2) )
      CVOMASS = 0.0
      CWMASS = 0.0
      AJXMASS = 0.0
```

```
      BJYMASS = 0.0
      AWMASS = 0.0
      CJXMASS = 0.0
      ELSEIF(MOD(M + P,2) .NE. 0 .AND. N .EQ. Q) THEN
      AWMASS = -1.0 * (
     1B*I5BAR*M*P/(P**2-M**2) )
      CJXMASS = -1.0 * (
     1-B*I5BAR*M*P/(P**2-M**2) )
      CVOMASS = 0.0
      CWMASS = 0.0
      AJXMASS = 0.0
      BJYMASS = 0.0
      BVOMASS = 0.0
      BWMASS = 0.0
      CJYMASS = 0.0
      ELSE
      BVOMASS = 0.0
      CVOMASS = 0.0
      AWMASS = 0.0
      BWMASS = 0.0
      CWMASS = 0.0
      AJXMASS = 0.0
      CJXMASS = 0.0
      BJYMASS = 0.0
      CJYMASS = 0.0
      ENDIF
C
      ELSE
C
C     BUCKLING PROBLEM - WE ARE LOOKING FOR THE CRITICAL BUCKLING
C     LOADS
      BVOMASS = 0.0
      CVOMASS = 0.0
      AWMASS = 0.0
      BWMASS = 0.0
      AJXMASS = 0.0
      CJXMASS = 0.0
      BJYMASS = 0.0
      CJYMASS = 0.0
      IF(M .EQ. P .AND. N .EQ. Q) THEN
      CWMASS = -1.0 * (
     1-B*P**2*PI**2/A/4.0 )
      ELSE
      CWMASS = 0.0
      ENDIF
C
      ENDIF
C--------------------------------------------------------------
C     STORE THESE TERMS IN THE MASS MATRIX
C--------------------------------------------------------------
      MASS(I,J) = AUOMASS
      MASS(I,J + MMAX * NMAX) = BUOMASS
```

```
      MASS(I,J + 2 * MMAX * NMAX) = CUOMASS
      MASS(I,J + 3 * MMAX * NMAX) = EUOMASS
      MASS(I,J + 4 * MMAX * NMAX) = GUOMASS
      MASS(I + MMAX * NMAX,J) = AVOMASS
      MASS(I + MMAX * NMAX,J + MMAX * NMAX) = BVOMASS
      MASS(I + MMAX * NMAX,J + 2 * MMAX * NMAX) = CVOMASS
      MASS(I + MMAX * NMAX,J + 3 * MMAX * NMAX) = EVOMASS
      MASS(I + MMAX * NMAX,J + 4 * MMAX * NMAX) = GVOMASS
      MASS(I + 2 * MMAX * NMAX,J) = AWMASS
      MASS(I + 2 * MMAX * NMAX,J + MMAX * NMAX) = BWMASS
      MASS(I + 2 * MMAX * NMAX,J + 2 * MMAX * NMAX) = CWMASS
      MASS(I + 2 * MMAX * NMAX,J + 3 * MMAX * NMAX) = EWMASS
      MASS(I + 2 * MMAX * NMAX,J + 4 * MMAX * NMAX) = GWMASS
      MASS(I + 3 * MMAX * NMAX,J) = AJXMASS
      MASS(I + 3 * MMAX * NMAX,J + MMAX * NMAX) = BJXMASS
      MASS(I + 3 * MMAX * NMAX,J + 2 * MMAX * NMAX) = CJXMASS
      MASS(I + 3 * MMAX * NMAX,J + 3 * MMAX * NMAX) = EJXMASS
      MASS(I + 3 * MMAX * NMAX,J + 4 * MMAX * NMAX) = GJXMASS
      MASS(I + 4 * MMAX * NMAX,J) = AJYMASS
      MASS(I + 4 * MMAX * NMAX,J + MMAX * NMAX) = BJYMASS
      MASS(I + 4 * MMAX * NMAX,J + 2 * MMAX * NMAX) = CJYMASS
      MASS(I + 4 * MMAX * NMAX,J + 3 * MMAX * NMAX) = EJYMASS
      MASS(I + 4 * MMAX * NMAX,J + 4 * MMAX * NMAX) = GJYMASS
C----------------------------------------------------------------
      J = J + 1
   20 CONTINUE
      I = I + 1
      J = 1
   10 CONTINUE
C----------------------------------------------------------------
C     CALL THE IMSL LIBRARY SUBROUTINE. USE THE MASS AND STIFFNESS
C     MATRICES AS INPUT AND FIND THE EIGENVALUES AND EIGENVECTORS.
      CALL DGVCRG(MSIZE,STIFF,MSIZE,MASS,MSIZE,ALPHA,BETA,EVEC,
     1MSIZE)
      DO 40 I = 1,MSIZE
      IF(BETA(I) .NE. 0.0) THEN
      EVAL(I) = ALPHA(I) / BETA(I)
      ELSE
      EVAL(I) = (1.0D+30 , 0.0D+00)
      ENDIF
   40 CONTINUE
      IF(NBUCVIB .EQ. 1) THEN
C
C     PRINT OUT THE FIRST 10 MODES FOR THE VIBRATION PROBLEM
C
      DO 50 I = 1,10
      REVAL = DREAL(EVAL(I))
      AGEVAL = DIMAG(EVAL(I))
      IF(ABS(AGEVAL) .GT. 1.0D-15) THEN
      WRITE(2,115) I
      ELSEIF(REVAL .GT. 1.0D+28) THEN
      WRITE(2,125) I
```

```
      ELSEIF(REVAL .LT. 0.0) THEN
      WRITE(2,120) I
      ELSE
      OMEGA = SQRT(REVAL)
      WRITE(2,130) I,REVAL,OMEGA
      ENDIF
   50 CONTINUE
C
      ELSE
C
C     PRINT OUT THE CRITICAL BUCKLING LOAD.  THE CRITICAL
C     BUCKLING LOAD IS THE EIGENVALUE WITH THE SMALLEST ABSOLUTE
C     VALUE.
C
      DO 55 I = 2,MSIZE
      IF(ABS(DIMAG(EVAL(I-1))) .GT. 1.0D-15) THEN
      GO TO 55
      ENDIF
      IF(ABS(DREAL(EVAL(I))) .GT. ABS(DREAL(EVAL(I-1))) .AND. ABS(
     1DREAL(EVAL(I-1))) .LT. 1.0D+28) THEN
      WRITE(2,220) DREAL(EVAL(I-1))
      ENDIF
   55 CONTINUE
C
      ENDIF
C
C     PRINT OUT THE 1ST MODE OF THE DEFLECTION, W(X,Y), ALONG THE
C     MIDLINES OF THE PANEL:  X = A/2 AND Y = B/2
C
C     PRINT OUT THE W EIGENVECTOR, CMN
C
      II = 1
      WRITE(2,500)
      WRITE(2,510)
      MNWMIN = 1 + 2 * MMAX * NMAX
      MNWMAX = 3 * MMAX * NMAX
      DO 400 I = MNWMIN, MNWMAX
      REVEC(II) = DREAL(EVEC(I,1))
      AGEVEC = DIMAG(EVEC(I,1))
      IF(ABS(AGEVEC) .GT. 1.0D-15) THEN
      WRITE(2,520) I,II,REVEC(II)
      ELSE
      WRITE(2,530) I,II,REVEC(II)
      ENDIF
      II = II + 1
  400 CONTINUE
C
C     DETERMINE W(X=A/2,Y)
C
      ASTEP = A / 50.0
      BSTEP = B / 50.0
      XCOORD = A / 2.0
```

```
      YCOORD = 0.0
      WRITE(2,540)
      WRITE(2,542)
  801 WMODE = 0.0
      JJJ = 1
      DO 470 M = 1,MMAX
      DO 472 N = 1,NMAX
      WMODE = WMODE + REVEC(JJJ)*SIN(M*PI*XCOORD/A)*SIN(N*PI*YCOOR
     1D/B)
      JJJ = JJJ + 1
  472 CONTINUE
  470 CONTINUE
      WRITE(2,550) YCOORD,WMODE
      YCOORD = YCOORD + BSTEP
      IF(YCOORD .GT. B) THEN
      GO TO 800
      ELSE
      GO TO 801
      ENDIF
C
  800 YCOORD = B / 2.0
C
C     DETERMINE W(X,Y=B/2)
C
      XCOORD = 0.0
      WRITE(2,560)
      WRITE(2,570)
  810 WMODE = 0.0
      JJJ = 1
      DO 480 M = 1, MMAX
      DO 482 N = 1, NMAX
      WMODE = WMODE + REVEC(JJJ)*SIN(M*PI*XCOORD/A)*SIN(N*PI*YCOOR
     1D/B)
      JJJ = JJJ + 1
  482 CONTINUE
  480 CONTINUE
      WRITE(2,550)XCOORD,WMODE
      XCOORD = XCOORD + ASTEP
      IF(XCOORD .GT. A) THEN
      GO TO 850
      ELSE
      GO TO 810
      ENDIF
C-------------------------------------------------------------
  115 FORMAT(/,9X,I3,11X,'EIGENVALUE IS COMPLEX')
  120 FORMAT(/,9X,I3,11X,'EIGENVALUE IS NEGATIVE')
  125 FORMAT(/,9X,I3,11X,'EIGENVALUE IS INFINITE')
  130 FORMAT(/,9X,I3,10X,D20.13,12X,D20.13)
  200 FORMAT(/,9X,I3,10X,D20.13)
  220 FORMAT(//,5X,'CRITICAL BUCKLING LOAD = ',1X,D20.13)
  500 FORMAT(//,5X,'W EIGENVECTOR, CMN, FOR 1ST MODE')
  510 FORMAT(//,5X,'M,N',10X,'CMN')
```

```
520 FORMAT(/,5X,I4,2X,I4,12X,D20.13,3X,'COMPLEX')
530 FORMAT(/,5X,I4,2X,I4,12X,D20.13)
540 FORMAT(//,5X,'DEFLECTION, W(X=A/2,Y)')
542 FORMAT(//,5X,'Y (IN.)',10X,'W(A/2,Y) (IN.)')
550 FORMAT(/,5X,F6.2,11X,E15.8)
560 FORMAT(//,5X,'DEFLECTION, W(X,Y=B/2)')
570 FORMAT(//,5X,'X (IN.)',10X,'W(X,B/2) (IN.)')
850 RETURN
    END
```

Appendix E: Generating the Galerkin Equations

This appendix gives an example for M=N=2 that shows how Subroutine "GALERK" generates the stiffness and mass/inertia matrices for simply supported and clamped boundary conditions. If M=N=2, each degree of freedom is approximated by four terms, and four equations are generated for each degree of freedom.

The following applies for simply supported boundary conditions. The procedure for the degree of freedom u_o is outlined as follows, starting with the four nested sums:

$$\sum_{p=1}^{2} \sum_{q=1}^{2} \sum_{m=1}^{2} \sum_{n=1}^{2}$$

Equation Number	p	q	m	n	Integration Case	Galerkin Equation for u_o
1	1	1	1	1	1	2.52
			1	2	none	0
			2	1	none	0
			2	2	2	2.57
2		2	1	1	none	0
			1	2	1	2.52
			2	1	2	2.57
			2	2	none	0
3	2	1	1	1	none	0
			1	2	2	2.57
			2	1	1	2.52
			2	2	none	0
4		2	1	1	2	2.57
			1	2	none	0
			2	1	none	0
			2	2	1	2.52

The same operations are carried out for v_o, w, ψ_x, and ψ_y using their associated Galerkin equations for simply supported boundary conditions. The equations are put into matrix format by forming a new column at each m and n cycle, and forming a new row at each p and q cycle. The resulting stiffness and mass/inertia matrices for this example are (20x20). The eigenvalues and eigenvectors are solved as shown in Eq (3.1)

The eigenvalue problem for the clamped boundary is formulated in a similar manner. For u_o the format is:

$$\sum_{p=1}^{2} \sum_{q=1}^{2} \sum_{m=1}^{2} \sum_{n=1}^{2}$$

Equation Number	p	q	m	n	Integration Case	Galerkin Equation For u_o
1	1	1	1	1	1	2.64
			1	2	2	2.69
			2	1	3	2.74
			2	2	4	2.79
2		2	1	1	2	2.69
			1	2	1	2.64
			2	1	4	2.79
			2	2	3	2.74
3	2	1	1	1	3	2.74
			1	2	4	2.79
			2	1	1	2.64
			2	2	2	2.69
4		2	1	1	4	2.79
			1	2	3	2.74
			2	1	2	2.69
			2	2	1	2.64

Bibliography

1. Bert, Charles W. and M. Kumar. "Vibration of Cylindrical Shells of Bimodulus Composite Materials," AIAA Journal: 147-154 (May 1981).

2. Beyer, William H. CRC Standard Mathematical Tables, 27. Boca Raton: CRC Press, Inc., 1984.

3. Bowlus, John A. The Determination of the Natural Frequencies and Mode Shapes For Anisotropic Laminated Plates Including the Effects of Shear Deformation and Rotary Inertia. MS Thesis, AFIT/GA/AA/85S-1. School of Engineering, Air Force Institute of Technology (AU), Wright-Patterson AFB OH, September 1985.

4. Bowlus, John A., A.N. Palazotto, and J.M. Whitney. "Vibration of Symmetrically Laminated Rectangular Plates Considering Deformation and Rotary Inertia," AIAA Journal, 25: 1500-1511 (Nov 1987).

5. Brush, Don O. and Bo O. Almroth. Buckling of Bars, Plates, and Shells. McGraw-Hill, 1975.

6. Bushnell, D. Computerized Buckling Analysis of Shells. Dordrecht, The Netherlands: Martinus Nijhoff Pub., 1985.

7. Chajes, Alexander. Principles of Structural Stability Theory. Englewood Cliffs, NJ: Prentice-Hall, Inc., 1974

8. Dennis, Capt Scott T. Large Displacement and Rotational Formulation for Laminated Cylindrical Shells Including Parabolic Transverse Shear. PhD Dissertation. School of Engineering, Air Force Institute of Technology (AU), Wright-Patterson AFB OH, June 1988.

9. Jones, Robert M. Mechanics of Composite Materials. New York: Hemisphere Publishing Corp, 1975.

10. Koiter, W.T. "A Consistent First Approximation in the General Theory of Thin Elastic Shells," Proc Sym on Theory of Thin Elastic Shells: 12-33 Amsterdam, North Holland, 1960.

11. Meirovitch, Leonard. Analytical Methods in Vibrations. New York: The MacMillan Company, 1967.

12. Mindlin, R.D. "Influence of Rotary Inertia and Shear on Flexural Motions of Isotropic Elastic Plates," Journ of Applied Mech, 18 1951.

13. Palardy, Real F. The Buckling and Vibration of Composite Plates Using the Levy Method Considering Shear Deformation and Rotary Inertia. MS Thesis, AFIT/GAE/AA/87D-16. School of Engineering, Air Force Institute of Technology (AU), Wright-Patterson AFB OH, December 1987.

14 Reddy, J.N. Energy and Variational Methods in Applied Mechanics. John Wiley and Sons, 1984.

15. Reddy, J.N. "A Simple Higher-Order Theory for Laminated Composite Plates," Journ of Applied Mech,51: 745-752 (Dec 1984)

16. Reddy, J.N. and C.F. Liu "A Higher-Order Shear Deformation Theory of Laminated Elastic Shells," Int J Eng Sci,23: 319-330 (Mar 1985)

17. Reddy J.N., and N.D. Phan "Stability and Vibration of Isotropic, Orthotropic, and Laminated Plates According to a Higher Order Shear Deformation Theory," Journ of Sound and Vibration,98(2): 157-170 (1985).

18. Reissner, E. "The Effect of Transverse Shear Deformation on the Bending Of Elastic Plates," Journ of Applied Mech. (1945)

19. Saada, Adel S. Elasticity: Theory and Applications. New York: Pergamon Press, 1974.

20. Shames, Irving H. and Clive L. Dym Energy and Finite Element Methods in Structural Mechanics. New York: McGraw Hill Book Company, 1985.

21. Soldatos, K.P. "Buckling of Axially Compressed Antisymmetric Angle-Ply Laminated Circular Cylindrical Panels According to a Refined Shear Deformable Shell Theory," PVP: 63-71, (1986)

22. Wang, Chi-Teh. Applied Elasticity. New York: McGraw-Hill Book Company, Inc, 1953.

23. Whitney, James M. "Buckling of Anisotropic Laminated Cylindrical Plates," AIAA Journ,22,11: 1641-1645 (Nov 1984)

24. Whitney, James M. Structural Analysis of Laminated Anisotropic Plates. Lancaster,Pennsylvania: Technomic Publishing Company,Inc,1987.

25. Vax Unix MACSYMA Reference Manual. Massachusetts Institute of Technology: Symbolics,Inc. (Oct 1985)

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Block 19 Abstract

An analytical study is conducted to determine the fundamental frequencies and critical buckling loads for laminated anisotropic circular cylindrical shell panels, including the effects of transverse shear deformation and rotary inertia, by using the Galerkin Technique. A linearized form of Sander's shell strain-displacement relations are derived, which include a parabolic distribution of transverse shear strains. The theory is valid for laminate thickness to radius ratios, h/R, of 1/5. Higher order constitutive relations are derived for the laminate. A set of five coupled partial differential equations of motion and boundary conditions are derived and then solved using the Galerkin Technique. Simply supported and clamped boundary conditions are investigated.

The Galerkin method is tested for convergence to exact solutions. Comparisons with Donnell shell solutions are conducted. The effects of transverse shear deformation and rotary inertia are examined by comparing the results with classical solutions, where applicable. The radius of curvature is varied to determine the effects of membrane and bending coupling.

It is found that the Galerkin Technique converges for all panel configurations investigated; additionally, it is found that buckling problems need more terms in the approximation than vibration problems to obtain proper convergence. The theory compares exactly with the Donnell solutions, which are valid up to h/R = 1/50. As expected, as length to thickness ratios are reduced, shear deformation effects significantly lower the natural frequencies and buckling loads. Analysis also shows that rotary inertia effects are very small. Finally, as h/R is varied from 0 (flat plate) to 1/5 (maximum limit), the frequencies and buckling loads increase due to membrane and bending coupling.